ON THE PLASTIC ZONE FORMATION NEAR A CRACK IN ANISOTROPIC BODY

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ABSTRACT

The plastic zone near the Mode I crack tip in anisotropic body is studied. Considered is the case of plane stress conditions. The linear tensor constitutive equations are used for the problem formulation. The solving equations are written in terms of displacement vector. By means of variable discretization they reduced to linear system of algebraic equations. The solution of this system is found by modified method of unknowns consecutive reduction that is generalized Gauss's method.
As the result the relations of plastic zone formations is obtained. In particular, the development characteristic of main plastic zone near the crack tip is studied. Also it is observed that the second plastic zone forms on the body boundary. The junction manner of the both plastic zones into one plastic zone is studied. The influence of crack length on size and shape of the plastic zone is shown.

Key words: anisotropic body, crack, plastic zone.

1. INTRODUCTION

Various crack models are widely used in mechanics of elastoplastic fracture. To state these models it is necessary to know sizes and form of the crack tip plastic zone. Therefore the solution of corresponding boundary problems are needed. In works [1-3] were founded (both analytically and numerically)solutions of a number of boundary problems for plane and antiplane strain conditions and also for the plane stress. However, all of its concern particularly the plastic zone near the crack in isotropic body. For now the plastic zone near the crack in anisotropic body is not studied enough. Only a few works is devoted to this problem from which can be remarked the work [4]. Solutions of several boundary problems under the plane strain condition were found in that work numerically.
This work is devoted to the study of the plastic zone near the crack tip in anisotropic body for the case of plane stress condition.

2. FORMULATION AND SOLUTION OF BOUNDARY PROBLEM

The components of displacement vector $\boldsymbol{u}$ are chosen as the main unknown variables.
Governing equations are derived using linear tensor constitutive equations relating the components of stress tensor $\boldsymbol{S}$ with the components of strain tensor $\boldsymbol{D}$ in the form [5]:

$$S^{\alpha\beta} = \frac{H}{Z} g^{\alpha\beta} + \sqrt{\frac{K - \frac{H^2}{Z}}{\Xi - \frac{E^2}{Z}}} \left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{E}{Z} g^{\alpha\beta} \right) \tag{1}$$

Here
$E = g^{\alpha\beta} D_{\alpha\beta}$, $Z = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} g^{\gamma\delta}$, $H = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} S^{\gamma\delta}$, $K = F_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}$, $\Xi = G^{\alpha\beta\gamma\delta} D_{\alpha\beta} D_{\gamma\delta}$.
The tensors of anisotropy ***F*** and ***G*** are mutually inverse tensors, i.e.

$$F_{\alpha\beta\gamma\delta} G^{\alpha\beta\varepsilon\zeta} = \delta_\gamma^\varepsilon \delta_\delta^\zeta \quad (\varepsilon, \zeta). \tag{2}$$

Here

$$\delta_\eta^\iota = \begin{cases} 1 \; (\eta = \iota) \\ 0 \; (\eta \neq \iota) \end{cases}.$$

The components of anisotropy tensor ***F*** are determined from experimentally found relations (in small region of initial state) of all components of strain tensor ***D*** from each component of stress tensor ***S*** .
If components of anisotropy tensor ***F*** can be expressed by two constants then in view of Eq. (2) the Eq. (1) turn to Hencky – Nadai's equations [6, 7].
Containing in Eq. (1) the invariants of stress tensor ***S*** and strain tensor ***D*** are related with each other.
It is supposed [8] that

$$H = E. \tag{3}$$

Thermodynamic analysis of Eq. (1) shows [9] the radical $\sqrt{K - \frac{H^2}{Z}}$ is one-valued function of radical $\sqrt{\Xi - \frac{E^2}{Z}}$:

$$\sqrt{K - \frac{H^2}{Z}} = \varphi\left(\sqrt{\Xi - \frac{E^2}{Z}}\right).$$

It is accepted that for $\sqrt{\Xi - \frac{E^2}{Z}} < \upsilon$

$$\sqrt{K - \frac{H^2}{Z}} = \sqrt{\Xi - \frac{E^2}{Z}}, \tag{4}$$

and for $\sqrt{\Xi - \frac{E^2}{Z}} \geq \upsilon$

$$\sqrt{K - \frac{H^2}{Z}} = \sqrt{\Xi - \frac{E^2}{Z}}\left[1 - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\right], \tag{5}$$

where $\tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right) = \dfrac{\sqrt{\Xi - \frac{E^2}{Z}} - \upsilon + p - \sqrt[3]{\sqrt{\left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2} - \frac{r}{2}} + \sqrt[3]{\sqrt{\left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2} + \frac{r}{2}}}{\sqrt{\Xi - \frac{E^2}{Z}}}$,

$$p = \frac{1}{3}\frac{b}{c}, \quad q = \frac{1}{3}\frac{b^2}{c^2} + \frac{1}{c}, \quad r = \frac{2}{27}\frac{b^3}{c^3} - \frac{1}{3}\frac{b}{c^2} - \frac{1}{c}\left(\sqrt{\Xi - \frac{E^2}{Z}} - \upsilon\right).$$

The constants b and c are to be determined from experimentally obtained relation between radicals $\sqrt{\Xi - \frac{E^2}{Z}}$ and $\sqrt{K - \frac{H^2}{Z}}$ so that for $\sqrt{K - \frac{H^2}{Z}} \geq \upsilon$ this relation can be approximated by polynomial

$$\sqrt{\Xi - \frac{E^2}{Z}} = \sqrt{K - \frac{H^2}{Z}} + b\,(\sqrt{K - \frac{H^2}{Z}} - \upsilon)^2 + c\,(\sqrt{K - \frac{H^2}{Z}} - \upsilon)^3.$$

In view of Eqs. (3) and (5) the Eq. (1) is written as

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \tilde{\varphi}\,(\sqrt{\Xi - \frac{E^2}{Z}})\left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{E}{Z}\, g^{\alpha\beta}\right) \tag{6}$$

If Eqs. (4) and (5) are valid the plasticity criterion has the form

$$\sqrt{\Xi - \frac{E^2}{Z}} = \upsilon. \tag{7}$$

If components of anisotropy tensor $\boldsymbol{F}$ can be expressed by two constants then accounting Eq. (2) the criterion (7) turns to Mises's criterion [10].

It is meant that the body is orthotropic with main directions are parallel to the axis of Cartesian coordinate system x^1, x^2, x^3.

In case of plane stress condition

$$S^{11} = S^{11}(x^1, x^2), \quad S^{12} = S^{12}(x^1, x^2), \quad S^{22} = S^{22}(x^1, x^2),$$

and

$$S^{13} = 0, \quad S^{23} = 0, \quad S^{33} = 0. \tag{8}$$

On the basis of equilibrium equations, Eqs. (8), (6), and Cauchy's relations it is obtained the second order differential equations with partial coordinates x^1, x^2 derivations of the displacement components u_1, u_2:

$$\begin{aligned} &\overset{F}{\mu}_{AA} \frac{\partial^2 u_1}{\partial x^1 \partial x^1} + \mu_{BB} \frac{\partial^2 u_1}{\partial x^2 \partial x^2} + (\overset{F}{\mu}_{AC} + \mu_{BB}) \frac{\partial^2 u_2}{\partial x^1 \partial x^2} = Q^1; \\ &\mu_{BB} \frac{\partial^2 u_2}{\partial x^1 \partial x^1} + \overset{F}{\mu}_{CC} \frac{\partial^2 u_2}{\partial x^2 \partial x^2} + (\mu_{BB} + \overset{F}{\mu}_{CA}) \frac{\partial^2 u_1}{\partial x^1 \partial x^2} = Q^2. \end{aligned} \tag{9}$$

Here

$$\begin{aligned} Q^1 &= \frac{\partial \overset{F}{T}_A}{\partial x^1} + \frac{\partial T_B}{\partial x^2}; \\ Q^2 &= \frac{\partial T_B}{\partial x^1} + \frac{\partial \overset{F}{T}_C}{\partial x^2}. \end{aligned}$$

Let's point that

$$\begin{aligned} \overset{F}{T}_A &= \tilde{\varphi}\,(\sqrt{\Xi - \frac{E^2}{Z}})\left[\overset{F}{\mu}_{AA} \frac{\partial u_1}{\partial x^1} + \overset{F}{\mu}_{AC} \frac{\partial u_2}{\partial x^2} - (1 - \xi_{AF})\frac{E}{Z}\right], \\ T_B &= \tilde{\varphi}\,(\sqrt{\Xi - \frac{E^2}{Z}})\mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right), \\ \overset{F}{T}_C &= \tilde{\varphi}\,(\sqrt{\Xi - \frac{E^2}{Z}})\left[\overset{F}{\mu}_{CA} \frac{\partial u_1}{\partial x^1} + \overset{F}{\mu}_{CC} \frac{\partial u_2}{\partial x^2} - (1 - \xi_{CF})\frac{E}{Z}\right]. \end{aligned}$$

Here

$$E = \frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2} + \frac{\partial u_3}{\partial x^3},$$

$$\Xi = \mu_{AA} \frac{\partial u_1}{\partial x^1}\frac{\partial u_1}{\partial x^1} + 2\mu_{AC} \frac{\partial u_1}{\partial x^1}\frac{\partial u_2}{\partial x^2} + \mu_{CC} \frac{\partial u_2}{\partial x^2}\frac{\partial u_2}{\partial x^2} +$$
$$+ \mu_{BB}\left(\frac{\partial u_1}{\partial x^2}\frac{\partial u_1}{\partial x^2} + 2\frac{\partial u_1}{\partial x^2}\frac{\partial u_2}{\partial x^1} + \frac{\partial u_2}{\partial x^1}\frac{\partial u_2}{\partial x^1} \right) +$$
$$+ 2\mu_{AF} \frac{\partial u_1}{\partial x^1}\frac{\partial u_3}{\partial x^3} + 2\mu_{CF} \frac{\partial u_2}{\partial x^2}\frac{\partial u_3}{\partial x^3} + \mu_{FF} \frac{\partial u_3}{\partial x^3}\frac{\partial u_3}{\partial x^3}.$$

Above it is accepted the following notation:

$$G^{1111} \equiv \mu_{AA}, \quad G^{1212} \equiv \mu_{BB}, \quad G^{1122} \equiv \mu_{AC}, \quad G^{2222} \equiv \mu_{CC},$$
$$G^{1133} \equiv \mu_{AF}, \quad G^{2233} \equiv \mu_{CF}, \quad G^{3333} \equiv \mu_{FF},$$
$$\frac{G^{1133}}{G^{3333}} \equiv \xi_{AF}, \quad \frac{G^{2233}}{G^{3333}} \equiv \xi_{CF},$$
$$G^{1111} - \frac{G^{1133}}{G^{3333}} G^{3311} \equiv \tilde{\mu}_{AA}, \quad G^{1122} - \frac{G^{1133}}{G^{3333}} G^{3322} \equiv \tilde{\mu}_{AC},$$
$$G^{2211} - \frac{G^{2233}}{G^{3333}} G^{3311} \equiv \tilde{\mu}_{CA}, \quad G^{2222} - \frac{G^{2233}}{G^{3333}} G^{3322} \equiv \tilde{\mu}_{CC}.$$

It is supposed that on body's boundary are given both the stress vector ***P*** and the displacement vector ***u****.
On the basis of boundary conditions written in components of stress vector ***P***, Eqs. (8), (6), and Cauchy's relations it is obtained the first order differential equations with partial coordinates x^1, x^2 derivations of displacement components u_1, u_2:

$$\left(\tilde{\mu}_{AA} \frac{\partial u_1}{\partial x^1} + \tilde{\mu}_{AC} \frac{\partial u_2}{\partial x^2} \right) n_1 + \mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right) n_2 = P^1 + R^1;$$
$$\mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right) n_1 + \left(\tilde{\mu}_{CA} \frac{\partial u_1}{\partial x^1} + \tilde{\mu}_{CC} \frac{\partial u_2}{\partial x^2} \right) n_2 = P^2 + R^2. \tag{10}$$

Here

$$R^1 = \tilde{T}_A n_1 + T_B n_2;$$
$$R^2 = T_B n_1 + \tilde{T}_C n_2.$$

The Eqs. (9) and (10) can be integrated with Ilyushin's method of consequent approximations [11]. Thus for the first approximation the quantities Q^1, Q^2 and R^1, R^2 must be set equal to zero and for each following approximation they are calculated on the basis of the values of components u_1, u_2 that where found at previous approximation step.
A rectangular body of small thickness containing a central crack is considered. The body symmetry axes coincide with axes x^1, x^2.
The components P^1, P^2 is given on the bottom and upper crack surfaces as well as on the body side surfaces. The components u_1^*, u_2^* is given on the bottom and upper body surfaces. The boundary conditions are symmetric with reference to axes x^1, x^2. Thus it is possible to consider only a quarter of the body (see fig. 1).

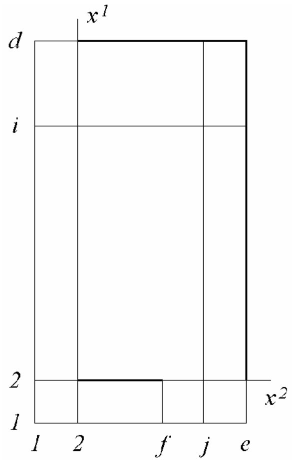

Figure 1. Considered part of the body.

On the upper crack surface ($n_1 = -1, \quad n_2 = 0$) the Eq. (10) has the form

$$-\left(\tilde{\mu}_{AA} \frac{\partial u_1}{\partial x^1} + \tilde{\mu}_{AC} \frac{\partial u_2}{\partial x^2} \right) = P^1 + R^1; \qquad -\mu_{BB} \left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right) = P^2 + R^2. \tag{11}$$

Here

$$R^1 = -\tilde{T}_A; \quad R^2 = -T_B.$$

On the side body surface ($n_1 = 0, \quad n_2 = 1$) the Eq. (10) has the form

$$\mu_{BB} \left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \right) = P^1 + R^1; \quad \tilde{\mu}_{CA} \frac{\partial u_1}{\partial x^1} + \tilde{\mu}_{CC} \frac{\partial u_2}{\partial x^2} = P^2 + R^2. \tag{12}$$

Here

$$R^1 = T_B; \quad R^2 = \tilde{T}_C.$$

On the upper body surface it is

$$u_1 = u_1^*; \quad u_2 = u_2^*. \tag{13}$$

As the result of symmetry with reference to axis x^1 it is

$$u_1(x^1, -x^2) - u_1(x^1, +x^2) = 0; \quad u_2(x^1, -x^2) + u_2(x^1, +x^2) = 0. \tag{14}$$

As the result of symmetry with reference to axis x^2 it is

$$u_1(-x^1, x^2) + u_1(+x^1, x^2) = 0; \quad u_2(-x^1, x^2) - u_2(+x^1, x^2) = 0. \tag{15}$$

Moreover the symmetry with reference to axis x^2 in the crack tip gives that

$$u_1 = 0; \quad \frac{\partial u_2}{\partial x^1} = 0. \tag{16}$$

The Eqs. (9), (11) – (16) are resolving equations of components u_1, u_2.
A plane grid of step h is created

$$\underset{i}{x^1} = (i-2)h \quad (i = 1, \dots, d), \quad \underset{j}{x^2} = (j-2)h \quad (j = 1, \dots, e).$$

Then it is noticed

$$u_1(\underset{i}{x^1}, \underset{j}{x^2}) \equiv y_s, \quad u_2(\underset{i}{x^1}, \underset{j}{x^2}) \equiv y_t. \tag{17}$$

Here

$$s = 2[(i-1)e + j - f] + 1, \quad t = 2[(i-1)e + j - f] + 2.$$

In view of Eq. (17) the coordinates $x^1,\ x^2$ partial derivations of components $u_1,\ u_2$ expressed through finite-differences and on the basis of Eqs. (9), (11) – (16) it is derived n linear algebraic equations $\left(n = 2\left(de - f + 1\right)\right)$ with unknowns $y_1,\dots,y_n$:

$$A_{s\,s}\, y_s + A_{s\,s+2e}\, y_{s+2e} + A_{s\,s-2e}\, y_{s-2e} + A_{s\,s+2}\, y_{s+2} + A_{s\,s-2}\, y_{s-2} +$$

$$+ A_{s\,t+2(e+1)}\, y_{t+2(e+1)} + A_{s\,t+2(e-1)}\, y_{t+2(e-1)} + A_{s\,t-2(e-1)}\, y_{t-2(e-1)} + A_{s\,t-2(e+1)}\, y_{t-2(e+1)} \approx B_s;$$

$$A_{t\,t}\, y_t + A_{t\,t+2e}\, y_{t+2e} + A_{t\,t-2e}\, y_{t-2e} + A_{t\,t+2}\, y_{t+2} + A_{t\,t-2}\, y_{t-2} +$$

$$+ A_{t\,s+2(e+1)}\, y_{s+2(e+1)} + A_{t\,s+2(e-1)}\, y_{s+2(e-1)} + A_{t\,s-2(e-1)}\, y_{s-2(e-1)} + A_{t\,s-2(e+1)}\, y_{s-2(e+1)} \approx B_t$$

$$\left(i = 2,\ j = f+1,\dots,e-1;\quad i = 3,\dots,d-1,\ j = 2,\dots,e-1\right);$$

$$A_{s\,s}\, y_s + A_{s\,s+2e}\, y_{s+2e} + A_{s\,s+4e}\, y_{s+4e} + A_{s\,t+2}\, y_{t+2} + A_{s\,t-2}\, y_{t-2} \approx B_s;$$

$$A_{t\,s+2}\, y_{s+2} + A_{t\,s-2}\, y_{s-2} + A_{t\,t}\, y_t + A_{t\,t+2e}\, y_{t+2e} + A_{t\,t+4e}\, y_{t+4e} \approx B_t$$

$$\left(i = 2,\ j = 2,\dots,f-1\right);$$

$$A_{s\,s}\, y_s + A_{s\,s-2}\, y_{s-2} + A_{s\,s-4}\, y_{s-4} + A_{s\,t+2e}\, y_{t+2e} + A_{s\,t-2e}\, y_{t-2e} \approx B_s;$$

$$A_{t\,s+2e}\, y_{s+2e} + A_{t\,s-2e}\, y_{s-2e} + A_{t\,t}\, y_t + A_{t\,t-2}\, y_{t-2} + A_{t\,t-4}\, y_{t-4} \approx B_t$$

$$\left(i = 2,\dots,d-1,\ j = e\right);$$

$$A_{s\,s}\, y_s = B_s;\quad A_{t\,t}\, y_t = B_t \qquad (18)$$

$$\left(i = d,\ j = 2,\dots,e\right);$$

$$A_{s-2\,s-2}\, y_{s-2} + A_{s-2\,s+2}\, y_{s+2} = B_{s-2};\quad A_{t-2\,t-2}\, y_{t-2} + A_{t-2\,t+2}\, y_{t+2} = B_{t-2}$$

$$\left(i = 2,\dots,d,\ j = 2\right);$$

$$A_{s-2e\,s-2e}\, y_{s-2e} + A_{s-2e\,s+2e}\, y_{s+2e} = B_{s-2e};\quad A_{t-2e\,t-2e}\, y_{t-2e} + A_{t-2e\,t+2e}\, y_{t+2e} = B_{t-2e}$$

$$\left(i = 2,\ j = f,\dots,e\right);$$

$$A_{s\,s}\, y_s = B_s;\quad A_{t\,t}\, y_t + A_{t\,t+2e}\, y_{t+2e} + A_{t\,t+4e}\, y_{t+4e} \approx B_t$$

$$\left(i = 2,\ j = f\right).$$

Here

$$-A_{s\,s} = 8\left(\tilde{\mu}_{AA} + \mu_{BB}\right),\quad A_{s\,s+2e} = 4\tilde{\mu}_{AA},\quad A_{s\,s-2e} = 4\tilde{\mu}_{AA},\quad A_{s\,s+2} = 4\mu_{BB},\quad A_{s\,s-2} = 4\mu_{BB},$$

$$A_{s\,t+2(e+1)} = \hat{\mu}_{AC} + \mu_{BB},\quad -A_{s\,t+2(e-1)} = \hat{\mu}_{AC} + \mu_{BB},\quad -A_{s\,t-2(e-1)} = \hat{\mu}_{AC} + \mu_{BB},$$

$$A_{s\,t-2(e+1)} = \hat{\mu}_{AC} + \mu_{BB},\quad B_s = 4h^2\, Q^1(\underset{i}{x^1},\ \underset{j}{x^2});$$

$$-A_{t\,t}=8\left(\mu_{BB}+\tilde{\mu}_{CC}\right),\quad A_{t\,t+2e}=4\mu_{BB},\quad A_{t\,t-2e}=4\mu_{BB},\quad A_{t\,t+2}=4\tilde{\mu}_{CC},\quad A_{t\,t-2}=4\tilde{\mu}_{CC},$$

$$A_{t\,s+2(e+1)}=\mu_{BB}+\tilde{\mu}_{CA},\quad -A_{t\,s+2(e-1)}=\mu_{BB}+\tilde{\mu}_{CA},\quad -A_{t\,s-2(e-1)}=\mu_{BB}+\tilde{\mu}_{CA},$$

$$A_{t\,s-2(e+1)}=\mu_{BB}+\tilde{\mu}_{CA},\quad B_t=4h^2\,Q^2(\underset{i}{x}^1,\underset{j}{x}^2)$$

$$\left(i=2,\ j=f+1,\ldots,e-1;\quad i=3,\ldots,d-1,\ j=2,\ldots,e-1\right);$$

$$A_{s\,s}=3\tilde{\mu}_{AA},\quad -A_{s\,s+2e}=4\tilde{\mu}_{AA},\quad A_{s\,s+4e}=\tilde{\mu}_{AA},\quad -A_{s\,t+2}=\tilde{\mu}_{AC},\quad A_{s\,t-2}=\tilde{\mu}_{AC},$$

$$B_s=2h[P^1(\underset{i}{x}^1,\underset{j}{x}^2)+R^1(\underset{i}{x}^1,\underset{j}{x}^2)];$$

$$-A_{t\,s+2}=\mu_{BB},\quad A_{t\,s-2}=\mu_{BB},\quad A_{t\,t}=3\mu_{BB},\quad -A_{t\,t+2e}=4\mu_{BB},\quad A_{t\,t+4e}=\mu_{BB},$$

$$B_t=2h[P^2(\underset{i}{x}^1,\underset{j}{x}^2)+R^2(\underset{i}{x}^1,\underset{j}{x}^2)]$$

$$(i=2,\ j=2,\ldots,f-1);$$

$$A_{s\,s}=3\mu_{BB},\quad -A_{s\,s-2}=4\mu_{BB},\quad A_{s\,s-4}=\mu_{BB},\quad A_{s\,t+2e}=\mu_{BB},\quad -A_{s\,t-2e}=\mu_{BB},$$

$$B_s=2h[P^1(\underset{i}{x}^1,\underset{j}{x}^2)+R^1(\underset{i}{x}^1,\underset{j}{x}^2)];$$

$$A_{t\,s+2e}=\tilde{\mu}_{CA},\quad -A_{t\,s-2e}=\tilde{\mu}_{CA},\quad A_{t\,t}=3\tilde{\mu}_{CC},\quad -A_{t\,t-2}=4\tilde{\mu}_{CC},\quad A_{t\,t-4}=\tilde{\mu}_{CC},$$

$$B_t=2h[P^2(\underset{i}{x}^1,\underset{j}{x}^2)+R^2(\underset{i}{x}^1,\underset{j}{x}^2)]$$

$$(i=2,\ldots,d-1,\ j=e);$$

$$A_{s\,s}=1,\quad B_s=u_1^*(\underset{i}{x}^1,\underset{j}{x}^2);\quad A_{t\,t}=1,\quad B_t=u_2^*(\underset{i}{x}^1,\underset{j}{x}^2)$$

$$(i=d,\ j=2,\ldots,e);$$

$$A_{s-2\,s-2}=1,\quad -A_{s-2\,s+2}=1,\quad B_{s-2}=0;\quad A_{t-2\,t-2}=1,\quad A_{t-2\,t+2}=1,\quad B_{t-2}=0$$

$$(i=2,\ldots,d,\ j=2);$$

$$A_{s-2e\,s-2e}=1,\quad A_{s-2e\,s+2e}=1,\quad B_{s-2e}=0;\quad A_{t-2e\,t-2e}=1,\quad -A_{t-2e\,t+2e}=1,\quad B_{t-2e}=0$$

$$(i=2,\ j=f,\ldots,e);$$

$$A_{s\,s}=1,\quad B_s=0;\quad -A_{t\,t}=3,\quad A_{t\,t+2e}=4,\quad -A_{t\,t+4e}=1,\quad B_t=0$$

$$(i=2,\ j=f).$$

Quantities $Q^1(\underset{i}{x}^1,\underset{j}{x}^2)$, $Q^2(\underset{i}{x}^1,\underset{j}{x}^2)$ and $R^1(\underset{i}{x}^1,\underset{j}{x}^2)$, $R^2(\underset{i}{x}^1,\underset{j}{x}^2)$ must be expressed in terms of unknown $y_1,\ldots,y_n$.

The solution of Eq. (18) is found with the method suggested in work [4].

The data of D16 alloy [5] is used.

The essential components of the anisotropy tensor ***F*** are

$$F_{1111}=0{,}193\cdot10^{-10}\,Pa^{-1},\quad -F_{1122}=0{,}045\cdot10^{-10}\,Pa^{-1},\quad -F_{1133}=0{,}049\cdot10^{-10}\,Pa^{-1},$$

$$F_{1212}=0{,}107\cdot10^{-10}\,Pa^{-1},\quad F_{1313}=0{,}121\cdot10^{-10}\,Pa^{-1},\quad F_{2222}=0{,}142\cdot10^{-10}\,Pa^{-1},$$

$$-F_{2233}=0{,}045\cdot10^{-10}\,Pa^{-1},\quad F_{2323}=0{,}107\cdot10^{-10}\,Pa^{-1},\quad F_{3333}=0{,}193\cdot10^{-10}\,Pa^{-1}.$$

The essential components of the anisotropy tensor ***G*** are

$$G^{1111} = 6{,}395 \cdot 10^{10}\ Pa, \quad G^{1122} = 2{,}744 \cdot 10^{10}\ Pa, \quad G^{1133} = 2{,}263 \cdot 10^{10}\ Pa,$$
$$G^{1212} = 2{,}336 \cdot 10^{10}\ Pa, \quad G^{1313} = 2{,}066 \cdot 10^{10}\ Pa, \quad G^{2222} = 8{,}781 \cdot 10^{10}\ Pa,$$
$$G^{2233} = 2{,}744 \cdot 10^{10}\ Pa, \quad G^{2323} = 2{,}336 \cdot 10^{10}\ Pa, \quad G^{3333} = 6{,}395 \cdot 10^{10}\ Pa.$$

The relation between radical $\sqrt{\Xi - \frac{E^2}{Z}}$ and radical $\sqrt{K - \frac{H^2}{Z}}$ is characterized by

$\upsilon = 3{,}25 \cdot 10^2\ Pa^{\frac{1}{2}}, \quad b = 0{,}1964347 \cdot 10^{-2}\ Pa^{-\frac{1}{2}}, \quad c = 0{,}5632820 \cdot 10^{-4}\ Pa^{-1}.$

It is accepted that

$$h = 2 \cdot 10^{-4}\ m,$$
$$d = 302, \quad e = 152, \quad f = 62,\ 42,\ 22.$$

For the given grid step *h* and parameter *f* the crack length *L* is taken the following values

$$2{,}4 \cdot 10^{-2}\ m, \quad 1{,}6 \cdot 10^{-2}\ m, \quad 0{,}8 \cdot 10^{-2}\ m.$$

The solution of the boundary problem is found for

$$P^1(\underset{i}{x}^1, \underset{j}{x}^2) = 0, \quad P^2(\underset{i}{x}^1, \underset{j}{x}^2) = 0$$
$$(i = 2,\ j = 2, \ldots, f - 1);$$

$$P^1(\underset{i}{x}^1, \underset{j}{x}^2) = 0, \quad P^2(\underset{i}{x}^1, \underset{j}{x}^2) = 0$$
$$(i = 2, \ldots, d - 1,\ j = 2);$$

$$u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) > 0, \quad u_2^*(\underset{i}{x}^1, \underset{j}{x}^2) = 0$$
$$(i = d,\ j = 2, \ldots, e).$$

The values of unknown $y_1, \ldots, y_n$ are found for nine steps of consequent approximations. Moreover, on the ninth step of the approximation the indexes *i* and *j* when the radical $\sqrt{\Xi - \frac{E^2}{Z}}$, that stands in the left part of criterion (7), gets grater or smaller then the constant υ are found. It allows to calculate the coordinates of the plastic zone boundary points.

3. RESULTS

The fig. 2 $(L = 2{,}4 \cdot 10^{-2}\ m)$, fig. 3 $(L = 1{,}6 \cdot 10^{-2}\ m)$ and the fig. 4 $(L = 0{,}8 \cdot 10^{-2}\ m)$ show the plastic zone formation.

The curves shown in fig. 2 $(L = 2{,}4 \cdot 10^{-2}\ m)$ are given for the following values of $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) \cdot 10^6, m$:

$$1 - 70,\ 2 - 80,\ 3 - 82,\ 4 - 84,\ 5 - 86,\ 6 - 88.$$

The plastic zone boundary for the case of plane strain condition and $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) = 70 \cdot 10^{-6}\ m$ marked in the figure by dashed line [4].

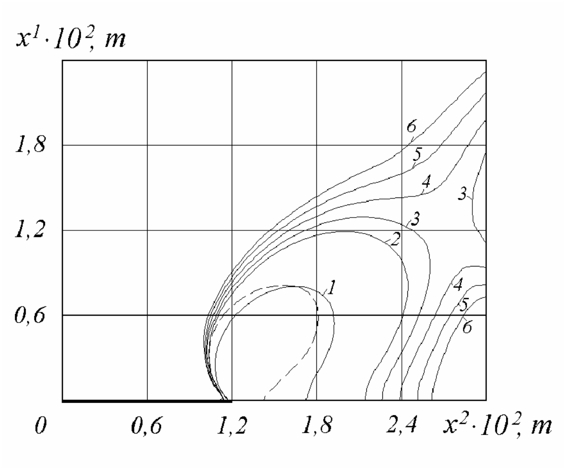

Figure 2. Plastic zone formation for $L = 2{,}4 \cdot 10^{-2} m$.

It should be noted that in comparison with the case of plane strain condition the main plastic zone near the crack tip under plane stress condition and $u_1^*(\underset{i}{x^1}, \underset{j}{x^2}) = 70 \cdot 10^{-6} m$ suffers some changes. Really, it significantly moves in x^2- axis direction and increases in sizes. So its length in x^2- axis direction increases in more then two times. The further behavior of the plastic zone remains without changes. Indeed, as the body elongates it expands and declines to the side surface of the body. For some value of the component $u_1^*(\underset{i}{x^1}, \underset{j}{x^2})$, that more then $80 \cdot 10^{-6} m$ and less then $82 \cdot 10^{-6} m$, the additional plastic zone appears on the side surface of the body. It locates in the region of the point ($\underset{70}{x^1} = 1{,}36 \cdot 10^{-2} m$, $\underset{152}{x^2} = 3{,}00 \cdot 10^{-2} m$). The further body elongation leads to the expansion of both plastic zones and its joint creating one plastic zone. It is observed the further expansion of the joint plastic zone. It should be noted that its form near the body side surface does change essentially.

The curves shown in fig. 3 ($L = 1{,}6 \cdot 10^{-2} m$) are given for the following values of $u_1^*(\underset{i}{x^1}, \underset{j}{x^2}) \cdot 10^6, m$:

1 – 70, 2 – 80, 3 – 88, 4 – 89, 5 – 90, 6 – 91, 7 - 92.

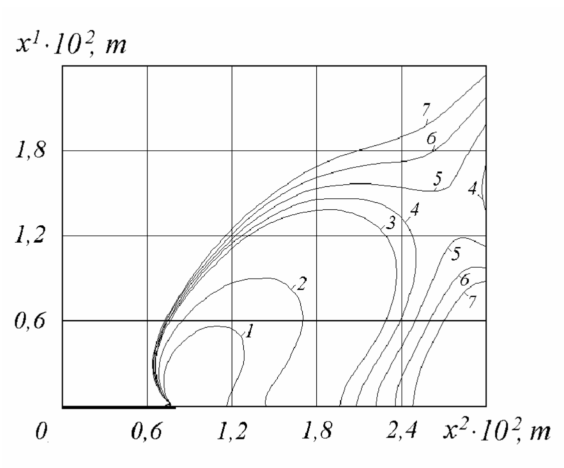

Figure 3. Plastic zone formation for $L = 1{,}6 \cdot 10^{-2} m$.

It is noticed that the main plastic zone arising near the crack tip for $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) = 70 \cdot 10^{-6} m$ gets considerably smaller.
As above with the body elongating, the main plastic zone expands and declines to the side surface of the body. For some value of the component $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2)$, that more then $88 \cdot 10^{-6} m$ and less then $89 \cdot 10^{-6} m$, the additional plastic zone appears on the side surface of the body. It locates at the vicinity of the point ($\underset{78}{x}^1 = 1{,}52 \cdot 10^{-2}\ m,\ \underset{152}{x}^2 = 3{,}00 \cdot 10^{-2}\ m$). The further body elongation leads to the expansion of both plastic zones and its joint creating one plastic zone. The further expansion of the joint plastic zone is observed. It is interesting that its form near the body side surface also does change essentially.
The curves shown in fig. 4 ($L = 0{,}8 \cdot 10^{-2} m$) are given for the next values of $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) \cdot 10^6, m$:

1 – 70, 2 – 80, 3 – 90, 4 – 92, 5 – 93, 6 – 94, 7 – 95, 8 - 96.

It is noticed that the main plastic zone arising near the crack tip for $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2) = 70 \cdot 10^{-6} m$ gets yet more smaller.
As the component $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2)$ increases, the main plastic zone expands and declines to the side surface of the body weaker. For some value of the component $u_1^*(\underset{i}{x}^1, \underset{j}{x}^2)$, that more then $94 \cdot 10^{-6} m$ and less then $95 \cdot 10^{-6} m$, the additional plastic zone appears on the side surface of the body. It locates in the region of the point ($\underset{75}{x}^1 = 1{,}46 \cdot 10^{-2}\ m,\ \underset{152}{x}^2 = 3{,}00 \cdot 10^{-2}\ m$). The further body elongation leads to the expansion of both plastic zones and its joint creating one plastic zone. The further expansion of the joint plastic zone is observed. It is important that its form near the body side surface does not change essentially.

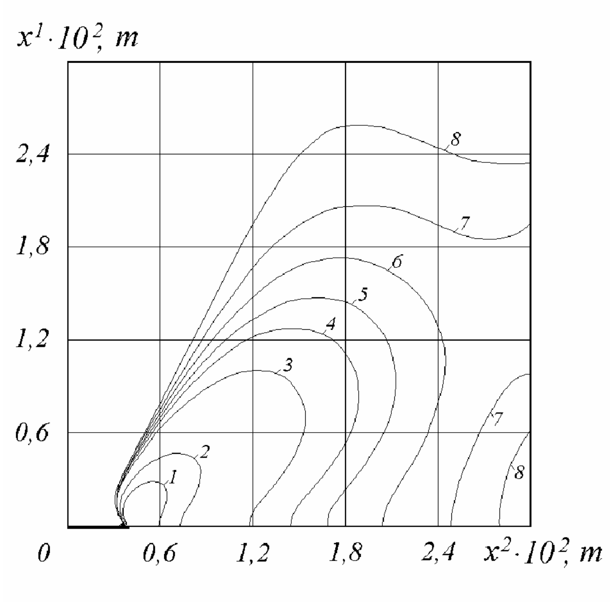

Figure 4. Plastic zone formation for $L = 0{,}8 \cdot 10^{-2} m$.

It is unsuspected that the point in which region an additional plastic zone arises weakly depends on the crack length *L*. Moreover with decreasing of the crack length *L* from $2{,}4\cdot 10^{-2}\,m$ to $1{,}6\cdot 10^{-2}\,m$ the coordinate $\underset{i}{x}^{1}$ of the point does increase from $1{,}36\cdot 10^{-2}\,m$ to $1{,}52\cdot 10^{-2}\,m$, and with decreasing of the crack length *L* from $1{,}6\cdot 10^{-2}\,m$ to $0{,}8\cdot 10^{-2}\,m$ it does decrease from $1{,}52\cdot 10^{-2}\,m$ to $1{,}46\cdot 10^{-2}\,m$.

4. CONCLUSIONS

Studied the influence of a Mode I crack length on the plastic zone formation with stiff loading of the body and under plane stress conditions. It was obtained that the crack length decreasing leads to the essential diminishing of the plastic zone near the crack tip and to the later appearance of additional plastic zone on the body side surface.

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