A SEMI-ANALYTICAL METHOD FOR SOLUTION OF THE LINEAR VISCOELASTICITY PROBLEMS

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ABSTRACT

Developing the ideas of work [1] considered is the problem of stress-strain state determination for linearly viscoelastic anisotropic body in case when the solution of the corresponding elastic problem can be found numerically. It is proposed to find viscoelastic solution using continued fraction approximation of the stress or strain at hand as a function of elastic constants. This approximation can be obtained numerically by any efficient numerical method (e.g. BEM). If only the Volterra principle is valid the solution can be found using resolvent operators' algebra. The possible error minimization is made via method of rational approximation by Stoer. An algorithm of the solution as well as estimation of the possible error is presented. The advantages of the proposed technique applied to contact and fracture problems of linear viscoelasticity are discussed.
As an example the solution of the stress concentration determination problem for orthotropic viscoelastic plane with rigid circular inclusion is given as well as the comparison with the earlier results obtained from the analytical solution.

Keywords: BEM, contact problem, continued fraction, fracture, integral operator, visoelasticity.

1. INTRODUCTION

Most of the works devoted to investigation of stressed state and fracture of linear viscoelastic bodies deal with the application of some techniques to obtain the solution of viscoelastic problem somehow reducing it to the elastic one [2,3,4].

In series of works by Prof. Kaminsky and his co-authors (see survey in [3]) an effective approach to solution of the problems of stressed state determination and study of crack growth in viscoelastic bodies was found (operator continued fraction method (OCFM)). This approach allows obtaining the solution of viscoelastic problem by expansion of viscoelastic solution (as a result of application the Volterra principle to the known analytical solution) into continued fraction of viscoelastic operators. The continued fraction can be easily reduced to linear combination of operators using resolvent operators algebra.

However, by now in a pure sense operator continued fraction method can be applied to solution of the problems with analytically solved elastic counterparts. This work is aimed to overcome this shortcoming and show the algorithm to solve viscoelastic problems which have numerical solution.

2. AN ALGORYTHM OF SOLUTION

2.1 Governing equations and boundary conditions

Consider linear viscoelastic orthotropic body of material with the following stress-strain interrelations

$$\varepsilon_{11}=\frac{1}{E_{11}^*}\sigma_{11}-\frac{\nu_{12}^*}{E_{22}^*}\sigma_{22};\ \varepsilon_{22}=-\frac{\nu_{21}^*}{E_{11}^*}\sigma_{11}+\frac{1}{E_{22}^*}\sigma_{22};\ \gamma_{12}=\frac{1}{G_{12}^*}\tau_{12} \tag{1}$$

where $\varepsilon_{11},\varepsilon_{22},\gamma_{12}$ are the strains, $\sigma_{11},\sigma_{22},\tau_{12}$ are the stresses in material and

$$\begin{gathered}\frac{1}{E_{11}^*}=\frac{1}{E_{11}^0}\left(1+\lambda_1R^*\left(\beta_1\right)\right);\ \frac{1}{E_{22}^*}=\frac{1}{E_{22}^0}\left(1+\lambda_2R^*\left(\beta_2\right)\right);\\ \frac{1}{G_{12}^*}=\frac{1}{E_{12}^0}\left(1+\lambda_3R^*\left(\beta_3\right)\right);\ \nu_{12}^*=\nu_{12}^0\left(1+\lambda_4R^*\left(\beta_4\right)\right);\nu_{21}^*=\nu_{21}^0\left(1+\lambda_5R^*\left(\beta_5\right)\right);\end{gathered} \tag{2}$$

where $E_{11}^0,E_{22}^0,G_{12}^0,\nu_{12}^0,\nu_{21}^0$ are instantaneous elastic modules of the material, λ_i,β_i are rheological parameters of the material,

$$R^*\left(\beta_i\right)\cdot f\left(t\right)=\int_0^t R(t-\tau,\beta_i)\,f\left(\tau\right)\mathrm{d}\tau \tag{3}$$

are Volterra's operators of the second order. According to the Boltzmann principle of linear viscoelasticity the other governing equation as well as the form of boundary conditions coincide with the elastic ones.

Hereafter we assume that boundary conditions meet the conditions of the Volterra principle applicability (at least for the stress or displacement that is crucial for strength of body), i.e., it suffices to replace elastic constants in solution of elastic problem by the corresponding viscoelastic operators (2) to determine viscoelastic solution. Denote the elastic solution as $\left(\sigma_{ij}\left(E_{11},E_{22},G_{12},\nu_{12},\nu_{21}\right),u_k\left(E_{11},E_{22},G_{12},\nu_{12},\nu_{21}\right)\right)$. Then the viscoelastic solution will be $\left(\sigma_{ij}^v\left(E_{11}^*,E_{22}^*,G_{12}^*,\nu_{12}^*,\nu_{21}^*\right),u_k^v\left(E_{11}^*,E_{22}^*,G_{12}^*,\nu_{12}^*,\nu_{21}^*\right)\right)$. The problem is reduce this solution to algebraic construction of operators from Eqs. (2).

2.2 Reducing the number of operators

As it was discussed in [1] the most general and convenient for the bounded operators are operators of Yu.N.Rabotnov which are resolvent, i.e.

$$\left(1+\lambda R^*\left(\beta\right)\right)^{-1}=1-\lambda R^*\left(\beta-\lambda\right). \tag{4}$$

This property can be used to reduce the number of operators in viscoelastic solution. Suppose that $\beta=\min\limits_i\beta_i$ then we can rewrite all operators in (2) via $R^*\left(\beta\right)$:

$$1+\lambda_iR^*\left(\beta_i\right)=1-\frac{\lambda_i}{\beta_i-\beta}+\frac{\lambda_i}{\beta_i-\beta}\cdot\frac{1}{1-\left(\beta_i-\beta\right)R^*\left(\beta\right)}. \tag{5}$$

Thus viscoelastic solution takes the form $\left(\sigma_{ij}^v\left(R^*\left(\beta\right)\right),u_k^v\left(R^*\left(\beta\right)\right)\right)$.

2.3 Reduction of viscoelastic solution as a classical Taylor series and approximate solution

Suppose that we have to find stress or displacement at the definite point of body. Let this solution is as follows $\varphi_m=F_m\left(R^*\left(\beta\right)\right)\cdot f\left(t\right)$, where F_m is analytic function determined by the structure of the elastic solution, f is the function that depends on the boundary conditions. According to work [5] in case of bounded viscoelastic operators F_m can be expanded into classical Taylor series

$$F_m\left(R^*(\beta)\right) = F_m(0) + F'_m(0)\cdot R^*(\beta) + \frac{1}{2!}F''_m(0)\cdot\left(R^*(\beta)\right)^2 + \ldots \qquad (6)$$

Thus, to obtain viscoelastic solution as Taylor series it suffices to know all derivatives of F_m at the zero point. This can be done by the investigation of the sequence elastic problems with proper variation of $R^*(\beta)$ as a real-value parameter x in Eqs.(2) and (5).

Now let we found an approximate solution of elastic counterpart of the problem at hand, say $\psi_m = \Phi_m\left(E_{11}, E_{22}, G_{12}, \nu_{12}, \nu_{21}\right)\cdot f$. Then corresponding function in viscoelastic solution has the form

$$\Phi_m\left(R^*(\beta)\right) = \Phi_m(0) + \Phi'_m(0)\cdot R^*(\beta) + \frac{1}{2!}\Phi''_m(0)\cdot\left(R^*(\beta)\right)^2 + \ldots \qquad (7)$$

Φ_m can be determined using finite differences. Suppose that we can obtain all coefficients in series (6) and (7) so that $\left|F_m^{(n)}\left(D^0_{ijkl}\right) - \Phi_m^{(n)}\left(D^0_{ijkl}\right)\right| < \varepsilon, n \in \mathbb{N}$. To estimate the accuracy of the approximate solution $\Phi_m\left(R^*(\beta)\right)$ we have (if $\left\|R^*(\beta)\right\| = \nu$):

$$\begin{aligned} \left\|F_m\left(R^*(\beta)\right) - \Phi_m\left(R^*(\beta)\right)\right\| &= \left\|\sum_{n=0}^{\infty}\frac{F_m^{(n)}(0) - \Phi_m^{(n)}(0)}{n!}\left(R^*(\beta)\right)^n\right\| \le \\ \le \sum_{n=0}^{\infty}\frac{\left|F_m^{(n)}(0) - \Phi_m^{(n)}(0)\right|}{n!}\left\|R^*(\beta)\right\|^n &< \varepsilon\sum_{n=0}^{\infty}\frac{\nu^n}{n!} = \varepsilon e^{\nu} \end{aligned} \qquad (8)$$

That means that for accuracy E we need to know all of derivatives of $F_m\left(R^*(\beta)\right)$ with the accuracy $E e^{\left\|R^*(\beta)\right\|}$ that is very strong restriction.

2.4 Continued fraction approach to the problem

Today numerical methods can give an approximate solution of elastic problem with the arbitrary precision (it depends only on physical capability of computer). Thus, we can neglect the possible error of approximate solution at the given point and consider the deviation of the approximate solution of viscoelastic problem as a whole. Stoer [6] proposed a method that in a reasonable turns of approximation can give an approximant to the function with given accuracy. This approximant is written in a form of continued fraction which is very convenient to calculations with resolvent operators. In our terms this will be

$$\Phi_m\left(R^*(\beta)\right) = e_0 + \cfrac{e_1\left(R^*(\beta) - x_1\right)}{1 + \cfrac{R^*(\beta) - x_2}{e_2 + \cfrac{R^*(\beta) - x_3}{e_3 + \ldots}}} \qquad (9)$$

where x_i are chosen on interval of approximation to minimize the possible error. The problem of possible error of approximation can not be solved so easy as it was for classical Taylor series because of this problem is incorrect as well as the problem for inverse Laplace transform for correspondence principle [1].

3. EXAMPLE

As an example we consider the problem of stress determination in infinite orthotropic viscoelastic plate with the rigid circular inclusion (Fig. 1). Rheological characteristics of the material is as follows

$$\begin{aligned} &E_{11}^0 = 23.0\,GPa, \lambda_1 = 0.0323\,s^{\alpha-1}, \beta_1 = -0.1570\,s^{\alpha-1}, \\ &E_{22}^0 = 16.0\,GPa, \lambda_1 = 0.1295\,s^{\alpha-1}, \beta_2 = -0.2745\,s^{\alpha-1}, \\ &G_{12}^0 = 3.08\,GPa, \lambda_3 = 0.0717\,s^{\alpha-1}, \beta_3 = -0.0276\,s^{\alpha-1}, \\ &\nu_{12}^0 = 0.11, \lambda_4 = \lambda_5 = \beta_4 = \beta_5 = 0, \alpha = 0.846, \\ &R(t-\tau,\beta) = \sum_{n=0}^{\infty} \frac{\beta^n (t-\tau)^{n(1-\alpha)-\alpha}}{\Gamma\left[(n+1)(1-\alpha)\right]} \end{aligned} \tag{10}$$

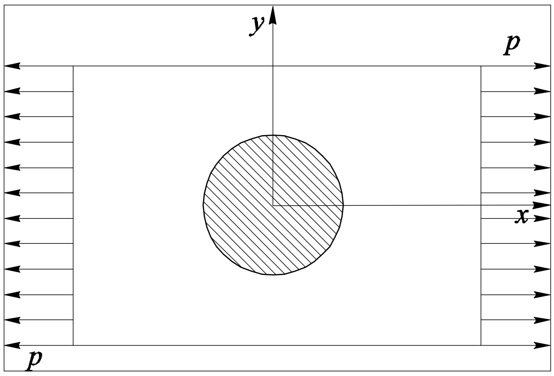

Figure1. Test problem geometry.

To obtain the approximate elastic solutions BEM in a form of fictive loadings is used [7]. According to this work displacements and stresses under symmetrical relative to origin constant loads P_x and P_y on interval of length $2a$, along $\bar{x}$-axis of rotated on angle β coordinates can be found as

$$\begin{aligned} u_x = &-\frac{1}{2\pi c_{66}(q_1-q_2)} P_x \left[\frac{\gamma_1}{q_1} I_1(\bar{x},\bar{y},\gamma_1) - \frac{\gamma_2}{q_2} I_1(\bar{x},\bar{y},\gamma_2) \right] - \\ &-\frac{1}{2\pi c_{66}(q_1-q_2)} P_y \left[I_2(\bar{x},\bar{y},\gamma_1) - I_2(\bar{x},\bar{y},\gamma_2) \right]; \\ u_y = &-\frac{1}{2\pi c_{66}(q_1-q_2)} P_x \left[I_2(\bar{x},\bar{y},\gamma_1) - I_2(\bar{x},\bar{y},\gamma_2) \right] - \\ &-\frac{1}{2\pi c_{66}(q_1-q_2)} P_y \left[\frac{q_1}{\gamma_1} I_1(\bar{x},\bar{y},\gamma_1) - \frac{q_2}{\gamma_2} I_1(\bar{x},\bar{y},\gamma_2) \right]; \end{aligned} \tag{11}$$

$$
\begin{aligned}
\sigma_{11} &= \frac{1}{2\pi(q_1-q_2)} P_x\left[\frac{1+q_1}{\gamma_1 q_1} I_3(\bar{x},\bar{y},\gamma_1) - \frac{1+q_2}{\gamma_2 q_2} I_3(\bar{x},\bar{y},\gamma_2)\right] + \\
&+ \frac{1}{2\pi(q_1-q_2)} P_y\left[\frac{1+q_1}{\gamma_1^2} I_4(\bar{x},\bar{y},\gamma_1) - \frac{1+q_2}{\gamma_2^2} I_4(\bar{x},\bar{y},\gamma_2)\right]; \\
\sigma_{22} &= -\frac{1}{2\pi(q_1-q_2)} P_x\left[\frac{\gamma_1(1+q_1)}{q_1} I_3(\bar{x},\bar{y},\gamma_1) - \frac{\gamma_2(1+q_2)}{q_2} I_3(\bar{x},\bar{y},\gamma_2)\right] - \\
&- \frac{1}{2\pi(q_1-q_2)} P_y\left[(1+q_1) I_4(\bar{x},\bar{y},\gamma_1) - (1+q_2) I_4(\bar{x},\bar{y},\gamma_2)\right]; \\
\sigma_{12} &= \frac{1}{2\pi(q_1-q_2)} P_x\left[\frac{1+q_1}{q_1} I_4(\bar{x},\bar{y},\gamma_1) - \frac{1+q_2}{q_2} I_4(\bar{x},\bar{y},\gamma_2)\right] - \\
&- \frac{1}{2\pi(q_1-q_2)} P_y\left[\frac{1+q_1}{\gamma_1} I_3(\bar{x},\bar{y},\gamma_1) - \frac{1+q_2}{\gamma_2} I_3(\bar{x},\bar{y},\gamma_2)\right],
\end{aligned}
\tag{12}
$$

where γ_1^2 and γ_2^2 are roots of bi-quadratic equation

$$c_{11}c_{66}\gamma^4 + \left[c_{12}(c_{12}+2c_{66}) - c_{11}c_{22}\right]\gamma^2 + c_{22}c_{66} = 0,$$

$$q_1 = \frac{c_{11}\gamma_1^2 - c_{66}}{c_{12}+c_{66}};\ q_2 = \frac{c_{11}\gamma_2^2 - c_{66}}{c_{12}+c_{66}};$$

$$c_{11} = \frac{D_{22}}{D_{11}D_{22}-D_{12}^2};\ c_{12} = -\frac{D_{12}}{D_{11}D_{22}-D_{12}^2};\ c_{22} = \frac{D_{11}}{D_{11}D_{22}-D_{12}^2},$$

$$D_{11} = \frac{1}{E_{11}};\ D_{12} = -\frac{\nu_{12}}{E_{11}};\ D_{22} = \frac{1}{E_{22}}$$

and I_k, $k=\overline{1,4}$ are complicated functions form work [7].

Taking into account the symmetry of the problem it suffices to study the state of quater of the inclusion, to apply BEM it was broken on 200 intervals (relative error of such a solution in elastic case is about 10^{-9}). To approximate viscoelastic function for stress along the loads we used Stoer approximation with error level $\lambda_1 = 10^{-5}$. The results obtaned here coincides with the results from work [8] which was obtained from the analytical solution of the problem using OCFM (Fig. 2).

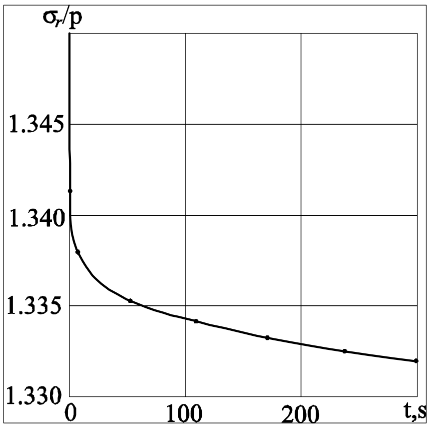

Figure2. Dimensionless radial stress along loading vs time (bullets shows analytical solution [8]).

4. CONCLUSIONS

In this report only a general framework of possible application of operator continued fraction method application to the numerically-solved problems of contact and fracture problems of viscoelasticity theory is given. However, it can be successfully applied to study stressed state of viscoelastic bodies.

To obtain the solution in example we use an efficient Stoer approximation. However, it must be applied with care in each case because of the problem incorrectness.

ACKNOWLEDGEMENTS

The author kindly thanks Prof. A.A.Kaminsky for his attention to this work.

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