THE DUGDALE'S TYPE CRACK MODEL FOR ELLIPTIC CRACK AT MULTIAXIAL LOADING

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ABSTRACT

The 3-D task for elliptic crack $\left(x^2/a^2+y^2/b^2\le 1\right)$ of mode I with plastic zone in plane $z=0$ is considered at uni-, bi- and triaxial loading in infinity. Material is suggested by ideally elastoplastic with yielding condition in general form. The plastic zone is modeled by elliptic ring with unknown sizes A, B along of the axis Ox and Oy. The stresses in the plastic zone are determined from yielding condition and condition of plane strain in local coordinate system connected with external contour of plastic zone. The following cases of loading are considered: 1) uniaxial loading in infinity $\sigma_z^\infty = p$; 2) biaxial loading $\sigma_z^\infty = p,\ \sigma_x^\infty = q\ \ \left(\sigma_y^\infty = g\right)$; 3) triaxial loading $\sigma_z^\infty = p,\ \sigma_x^\infty = q,\ \ \sigma_y^\infty = g$. The maximum value of stresses σ_z^0 in plastic zone is analyzed in dependence from sign and values of stresses q, g at von Mises yielding condition. In the first and third cases at $q=g$ the sizes A, B of the plastic zone are obtained in analytical form. The limit cases of small-scale yielding and fully plastic state are considered too.

Keywords: elliptic crack, plastic zone, multiaxial loading, plastic constraint factor.

1.INTRODUCTION

The Dugdale crack model [1] was suggested to thin plates (plane stress) *only*. Direct application of this model to plane strain or 3-D problem is not correct due to triaxial stress state at the crack front. Guo [2] and Neimitz [3] used a semi-analytical method to rationalize effects of out-of-plane and in-plane constraints for through- thickness cracked bodies. Analysis in these works is focused on small- scale yielding (SSY) conditions in which the crack-tip plastic zone is fully enclosed by K or $K-T$ dominated elastic field.

Galatenko proposed the Dugdale-type crack model to plane strain [4] and to circular disk-shaped crack [5] without limitations on the plastic zone sizes. Dependence of the plastic constraint factor (PCF) m on external loads level and loads acting parallel to the crack plane was obtained for Treska and von Mises yielding conditions. At SSY conditions the PCF ($m \equiv m_k$) is limited by imposed restrictions on the *T*-stresses.

The report presented deals with extension of the proposed crack model to elliptic crack. The results are compared with those for circular disk-shaped crack.

2. MODEL OF THE ELLIPTIC CRACK

Let us consider 3-D volume with elliptic crack $S\left\{S: x^2/a^2+y^2/b^2\le 1\right\}$ located in plane *z=0*, under acting of the triaxial stresses at infinity $\sigma_z^\infty = p,\ \sigma_x^\infty = q,\ \ \sigma_y^\infty = g$. The stresses *p* are tensional and *q, g* can be as tensional as compression. The material is assumed to be elastic-perfectly plastic with yielding condition

$$F\left(\sigma_1,\sigma_2,\sigma_3\right)=\sigma_Y, \tag{1}$$

where $\sigma_1,\sigma_2,\sigma_3$ are principal stresses, σ_Y is a yield stress.
The plastic zone ΔS at the crack front is modeled by elliptic ring in crack plane with unknown sizes *A, B* of the external contour. In the plastic zone the stresses $\sigma_z^0,\sigma_x^0,\sigma_y^0,\tau_{xy}^0\left(\tau_{xz}^0=\tau_{yz}^0=0\right)$ satisfy to yielding condition (1). After subtraction of the applied stress state $\sigma_z^\infty = p,\ \sigma_x^\infty = q,\ \ \sigma_y^\infty = g$ the auxiliary task can be obtained for plastic zone ΔS

$$z=0,\quad (x,y)\in\Delta S:\ \ \sigma_z=\sigma_z^0-p,\ \sigma_x=\sigma_x^0-q,\ \sigma_y=\sigma_y^0-g,\ \tau_{xy}=\tau_{xy}^0; \tag{2}$$

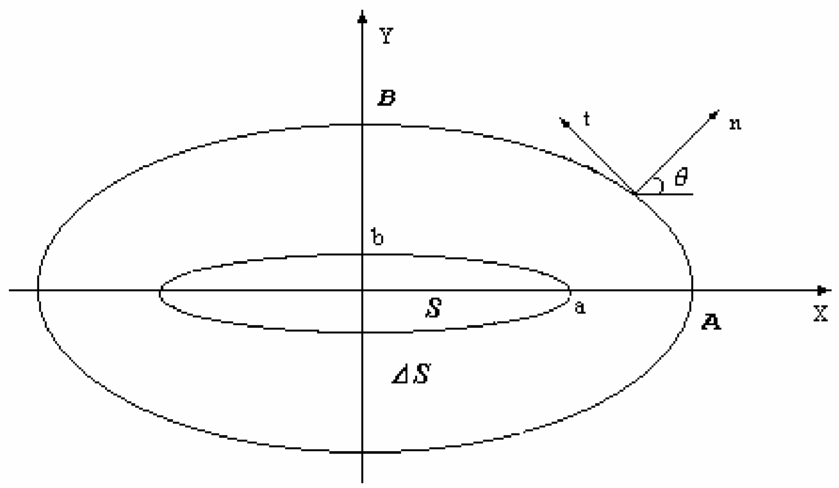

Figure 1.

In the local orthogonal coordinate system *ntz* (Fig.1), located at the external plastic zone contour the expression (2) have the form

$$z=0,\quad (x,y)\in\Delta S:\ \ \sigma_z=\sigma_z^0-p,\ \sigma_n=\sigma_n^0-\sigma_n^{(1)},\ \sigma_t=\sigma_t^0-\sigma_t^{(1)},\ \tau_{nt}=\tau_{nt}^0-\tau_{nt}^{(1)}. \tag{3}$$

Here $\sigma_n^{(1)},\sigma_t^{(1)},\tau_{nt}^{(1)}$ are stresses in plane $z=0$ without crack which are connected with $\sigma_x^\infty,\sigma_y^\infty$ by relations

$$\sigma_n^{(1)} = \frac{\sigma_x^\infty + \sigma_y^\infty}{2} + \frac{\sigma_x^\infty - \sigma_y^\infty}{2}\cos 2\theta;$$
$$\sigma_t^{(1)} = \frac{\sigma_x^\infty + \sigma_y^\infty}{2} - \frac{\sigma_x^\infty - \sigma_y^\infty}{2}\cos 2\theta;$$
$$\tau_{nt}^{(1)} = -\frac{\sigma_x^\infty - \sigma_y^\infty}{2}\sin 2\theta,$$

where θ is angle between normal *n* and axis *Ox*.
It is well known [6] that asymptotic stress field in the coordinate system *ntz* corresponds to plane strain

$$\sigma_z = \sigma_n,\ \sigma_t = \nu(\sigma_z + \sigma_n) = 2\nu\,\sigma_z,\quad \tau_{nt} = 0 \ \text{at}\ z = 0,\ r \to 0. \tag{4}$$

The stresses are limited and continuous on the boundary of elastic and plastic zones. Then from (3),(4) we have

$$\sigma_z^0 - p = \sigma_n^0 - \sigma_n^{(1)};\ \sigma_t^0 - \sigma_t^{(1)} = 2\nu(\sigma_z^0 - p);\ \tau_{nt}^0 - \tau_{nt}^{(1)} = 0. \tag{5}$$

The yielding condition (1) and relations (5) substitute the full equations system to determine the stresses in plastic zone. In particular, for von Mises yielding condition the obtained system have the form

$$\begin{cases} \sigma_z^0 - p = \sigma_n^0 - \sigma_n^{(1)};\ \sigma_t^0 - \sigma_t^{(1)} = 2\nu(\sigma_z^0 - p);\ \tau_{nt}^0 = \tau_{nt}^{(1)}; \\ (\sigma_z^0 - \sigma_n^0)^2 + (\sigma_n^0 - \sigma_t^0)^2 + (\sigma_t^0 - \sigma_z^0)^2 + 6\tau_{nt}^{02} = 2\sigma_Y^2. \end{cases} \tag{6}$$

After calculating of stresses in the plastic zone the boundary task of elasticity theory for crack $S + \Delta S$ can be formulated

$$\begin{array}{ll} & (x,y)\in S: \qquad \sigma_z = -p; \\ z = 0, & (x,y)\in \Delta S: \quad \sigma_z = \sigma_z^0 - p,\ . \\ & (x,y)\notin S + \Delta S: \quad u_z = 0. \end{array} \tag{7}$$

The further solution of problem (7) is similar to the Leonov-Panasyuk-Dugdale problem [7].
The stresses σ_z^0 or PCF $m = \sigma_z^0/\sigma_Y$ characterizes constraint effects on the crack front.
The solution (7) gives the following formula for σ_z^0

$$\sigma_z^0 = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + \frac{q+g}{2} - \frac{3}{2}(q-g)\cos 2\theta + \sqrt{4\sigma_Y^2 - 3(p - \frac{q+g}{2} - \frac{q-g}{2}\cos 2\theta)^2 - 3(q-g)^2\sin^2 2\theta}\right].. \tag{8}$$

In according to (8) the stresses σ_z^0 depend from angle θ. The extremum points can be obtained from the necessary condition $d\sigma_z^0/d\theta = 0$ which gives the following equation

$$(q-g)\sin 2\theta\left[\sqrt{4\sigma_Y^2 - 3(p - \frac{q+g}{2} - \frac{q-g}{2}\cos 2\theta)^2 - 3(q-g)^2\sin^2 2\theta} - p + \frac{q+g}{2} - \frac{3}{2}(q-g)\cos 2\theta\right] = 0. \tag{9}$$

Equation (9) has a solution at three cases:
1) $q = g,\ p \neq 0$. This case corresponds to axisymmetric loading when σ_z^0 independent of θ and has the form

$$\sigma_z^0 = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + q + \sqrt{4\sigma_Y^2 - 3(p-q)^2}\right]. \tag{10}$$

In particular, at $q = g = 0,\ p \neq 0$ we have uniaxial loading $\sigma_z^\infty = p$ at which

$$\sigma_z^0 = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + \sqrt{4\sigma_Y^2 - 3p^2}\right]. \tag{11}$$

Results at these schemes of loading are analyzed in paper [8].

2) Condition

$$\sqrt{4\sigma_T^2 - 3(p - \frac{q+g}{2} - \frac{q-g}{2}\cos 2\theta)^2 - 3(q-g)^2\sin^2 2\theta} - p + \frac{q+g}{2} - \frac{3}{2}(q-g)\cos 2\theta = 0, \tag{12}$$

after transformations, can be written as

$$(q-g)^2 + (g-p)^2 + (p-q)^2 = 2\sigma_Y^2. \tag{13}$$

The last equation corresponds to fully ductile fracture because the external loads satisfy to yielding condition.

3) $\sin 2\theta = 0$. The critical points $\theta_\bullet = \pi k/2 \quad (k = 0,1,2,3)$ can be obtained which coincide with ends of the axes of elliptic crack.

We consider the third case at bi- and triaxial nonaxisymmetric $(q \neq g)$ loads on infinity.

3. RESULTS AND DISCUSSION

3.1. Biaxial loading

1. At $\sigma_z^\infty = p,\ \sigma_x^\infty = q,\quad \sigma_y^\infty = 0$ the distribution σ_z^0 are determined by formulas

$$\sigma_z^0 = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + \frac{q}{2} - \frac{3}{2}q\cos 2\theta + \sqrt{4\sigma_T^2 - 3(p - \frac{q}{2} - \frac{q}{2}\cos 2\theta)^2 - 3q^2\sin^2 2\theta}\right].$$

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(13)

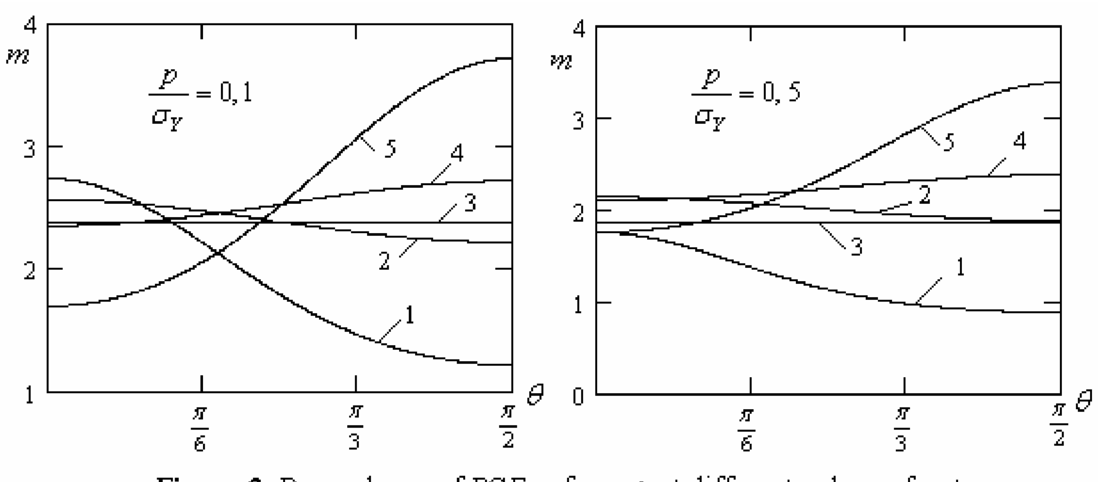

Figure 2. Dependence of PCF m from θ at different values of q/σ_Y

Fig.2 shows the dependence of PCF m on angle θ at different stresses q/σ_Y and $\nu = 0,3$. The curves 1,2,3,4,5 correspond to values $q/\sigma_T = -0,5; -0,1; 0; 0,1; 0,5$.

For given load the maximum values equal

$$\sigma^0_{z\max} = \sigma^0_z(\theta = 0) = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p - q + \sqrt{4\sigma_Y^2 - 3(p-q)^2}\right], \quad q < 0;$$

$$\sigma^0_{z\max} = \sigma^0_z(\theta = \pi/2) = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + 2q + \sqrt{4\sigma_Y^2 - 3p^2}\right], \quad q > 0. \tag{14}$$

In particular, at $q = \nu p$ we have from the second expression of (14)

$$\sigma^0_{z\max} = \sigma^0_z(\theta = \pi/2) = \frac{p}{2} + \frac{\sqrt{4\sigma_T^2 - 3p^2}}{2(1-2\nu)}. \tag{15}$$

The result (15) corresponds to plane strain case [4] for central crack $2b$.

2. At $\sigma_z^\infty = p \quad \sigma_y^\infty = g$ we have the following distribution

$$\sigma^0_z = \frac{1}{2(1-2\nu)}\left[(1-4\nu)p + \frac{g}{2} + \frac{3}{2}g\cos 2\theta + \sqrt{4\sigma_T^2 - 3(p - \frac{g}{2} + \frac{g}{2}\cos 2\theta)^2 - 3g^2\sin^2 2\theta}\right].. \tag{16}$$

Dependence m on angle θ is shown on Fig.3. Maximum m takes place on the ends of larger axis of ellipse at $g > 0$ and on the ends of smaller axis at $g < 0$. When $g = \nu p$, the ends of larger axis of elliptic crack have the same constraint that at plane strain.

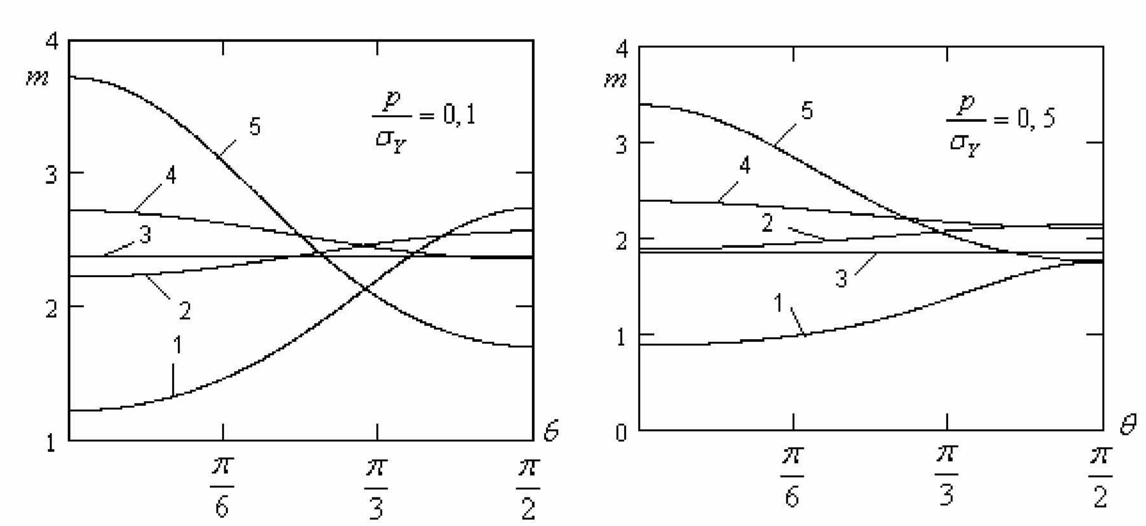

Figure 3. Dependence m on θ at diffirent g/σ_Y

3.1. Triaxial nonaxisymmetric loading

In general case of triaxial loading the distribution σ^0_z is determined by formulas (8). Triaxial loading on infinity can be presented as superposition of axisymetric (*q=g)* state and uniaxial tension or compression along of one axis of elliptic crack. The first state doesn't influence on the position of critical points. Consequently, their position is determined by the second state. Fig.4 shows $m-\theta$ dependence at values of stresses acted along of elliptic axes : 1 -

$q = 0,5\sigma_T$, $g = 0,1\sigma_T$; 2 - $q = 0,1\sigma_T$, $g = 0,5\sigma_T$; 3 - $q = 0,5\sigma_T$, $g = -0,1\sigma_T$; 4 - $q = -0,5\sigma_T$, $g = 0,1\sigma_T$; 5 - $q = g = 0$.

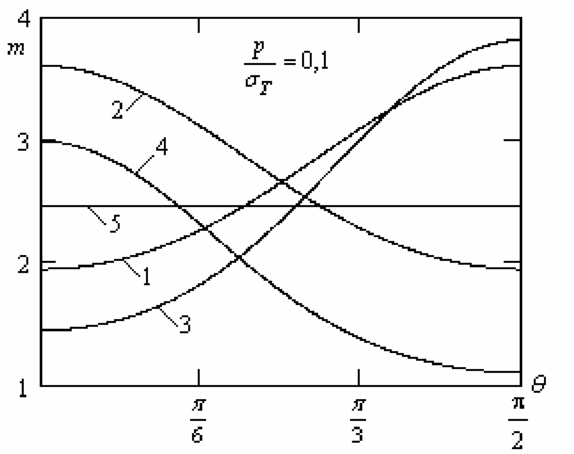

Figure 4

In the first and third schemes of loads maximum m takes place at $\theta = \pi/2$. When $g > q$ then maximum m is realized at $\theta = 0$.

4. **CONCLUSIONS**

Analyzing the results, we can formulate a rule for determination of the maximum constraint points of plastic strains at the elliptic crack front under multiaxial loading: the maximum constraint (maximum σ_z^0) occurs at those points on the crack periphery where the external tangential tensile stress are maximum. It means that in dependence on stresses q, g, acted along of elliptic crack axes, the fracture beginning can be on the ends of larger axis in spite of minimum of stress intensity factor in these points.

REFERENCES

1. Dugdale, D.S. "Yielding of steel sheets containing slits", **J. Mech. and Phys. of Solids,V**8, N2,1960.
2. Guo,W. "Three-dimensional analyses of plastic constraint for through-thickness cracked bodies", **Eng.Fract.Mech.,** 62, 1999.
3. Neimitz, A., "Dugdale model modification due to the geometry induced plastic constraints", **Eng.Fract.Mech.,** 67, 2000.
4. Galatenko, G.V., "To elastoplastic crack model at plane strain", **Prickladnay Mechanika,** V.28,N9, 1992.
5. Galatenko, G.V., "Circular disk-shaped crack at triaxial loading", **Prickladnay Mechanika,** V.28,N8, 1992.
6. Cherepanov, G.P., **Mechanics of brittle fracture** [in Russian], Nauka Publishing, Moscow, 1974.
7. Panasyuk, V.V., **The limiting state of brittle bodies with cracks** [in Russian], Naukova Dumka Publishing, 1968.
8. Galatenko, G.V., "A ring-shaped plastic zone at an elliptic crack under triaxial axisymmetric loading", **Prickladnaya Mechanika,** V.40,N2, 2004