FRACTURE UNDER INITIAL STRESSES ACTING ALONG CRACKS: APPROACHES, CONCEPTS AND RESULTS

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ABSTRACT

In this study the approach to studying the problems of fracture under initial stresses acting along cracks proposed by Guz' (1983) is expounded. Other approaches and concepts the problem concerned are briefly discussed. Some author's results for isolated and near-the-surface cracks are given below.

Key words: Initial stresses, fracture, fracture criteria, cracks.

1. INTRODUCTION

Under consideration is the problem of fracture of solids caused by initial stresses acting parallel to the crack surfaces. As well known this problem principally can't be modeled in the framework of classical linear fracture mechanics with classical fracture criteria Irwin-Griffith type or critical crack opening criterion because of simply missing stress components acting along the cracks in the criteria mentioned (for simplicity, the pure cases of the homogeneous states in the solids with system of parallel cracks caused by initial stresses parallel crack surfaces are investigated). Therewith it is intuitively obvious that initial stresses may appreciably influence the process of fracture (follow for example the 'visualized' experiments concerning the separating of initially stressed bars or stretched strings: see Guz' (1983)).

An analytical approach based on the relations of the three-dimensional linearised solid mechanics was presented by Guz' (1980, 1983, see also 1999)). Detailed showing of the concepts and results in the framework of this approach is through the whole study from the next section. Below in this section we only briefly mention another approaches and concepts concerning the problem.

In a large scale, another approaches to include initial stresses parallel to the crack surfaces involves or using non-brittle (plastic) fracture because of entering the corresponding stress components in an yield condition or constructing on the basis of classical linear theory fracture as named approximate (oversimplify, estimated) schemes.

As for plastic fracture, the comprehension of the situation may be acquired from every capital study on the subject. We mark only that obtaining specific results in this case is very difficult taking into account the very complexity of the mathematical side of the problem.

An approximate schemes are known long (away back in the thirties of the past century, see for example Obreimoff I.W. (1930)). In these schemes the stress components parallel to the cracks surfaces is introduced in the consideration rather artificially (for example, is introduced in the boundary conditions, etc.) on the base of some assumptions, which are not derived from strict equations for elastic body. Anyhow, the results obtained in such manner need the verification.

2. APPROACHES AND CONCEPTS

When in the body initial stresses parallel to the cracks surfaces are acting two classes of fracture problems are separated depending on there is or no the additional rather small stress field (in comparison with basic initial stress field, see Guz' (1983)).

In the absence of such additional stress field the class of fracture problems named as *the problems of fracture mechanics under compression along the cracks* is under consideration. The *approach* for studying of these problems is based on applying of relations of the three-dimensional linearised solid mechanics. The *concept* of fracture is : the beginning of the fracture process is determined by the mechanism of the local instability near the cracks. In the moment when the initial stresses are reaching their critical values (as values corresponding the local instability) the process of fracture is initiating. Details are given in the book Guz' A.N., Dyshel M.Sh. and Nazarenko V.M. (1992) or review Guz' A.N., Nazarenko V.M. (1989a, 1989b).

The present study is devoted to the class of fracture problems named as *the problems of the brittle fracture of solids with the initial stresses* acting along the cracks surfaces. In this case we have an additional stress field small in comparison with the basic initial stress field.

The ***approach*** is developed on the basis of relations of the three-dimensional linearised solid mechanics (Guz' A.N. (1983, 1992) presented general formulations of fracture mechanics problems with respect to the effect of the initial stresses). Also fracture criteria of Griffith-Irwin type were constructed. These criteria formulate in substance the ***concept*** of fracture in this case: the fracture process is in progress (crack is growing) if the certain combination of integral parameters of additional stress field near the crack tips (namely, stress intensity factors) is reaching its critical value. It should be noted that the stress intensity factors depend on the initial stresses values.

From the point of view of the mathematical apparatus used the following basic steps were made. *Firstly,* we are premising from the equations of geometrically nonlinear elasticity theory (Lurie A.I. (1990)). The field of initial stresses satisfies the equations mentioned. *Secondly,* we consequently derive the linearised relationships (geometrical ones, equation of motion, boundary conditions, stress-strain elasticity conditions) from the corresponding relationships of nonlinear elasticity. The values of initial stresses enter the linearised relationships as the coefficients. *In the third,* we consider initial stresses acting along the crack surfaces so that the initial stress state is homogeneous (we also consider as generally received the crack to be the mathematical cut). For the homogeneous initial stress state we used the common solutions of linearised equations somewhat similar to common solutions for the anisotropy (namely, transverse isotropy) body in the linear elasticity. And, *in forth,* for the formulated boundary problems we use the analytical functions apparatus for plane problems or the integral transformations apparatus for space problems.

3. PROBLEMS FORMULATIONS

Now we formulate problems for two important cases, namely, for isolated crack and for near the surface crack.
As an example, for isolated crack we formulate the problem for the *normal-rapture crack for the plane deformation case* (Guz' A.N. (1983)).

Let us consider indefinite space containing the crack of the length $2a$ which is situated in the plane y_1Oy_3 and is infinite in the Oy_3 direction ($|y_1| \le a; y_2 = \pm 0; -\infty < y_3 < +\infty$). The subscripts '+' and '-'in the expression $y_3 = \pm 0$ denote, respectively, the upper and lower crack surfaces. Consider the plane y_1Oy_2. The normal load on the crack boundaries is symmetric, so on the boundary conditions for the bottom halfspace $y_2 \le 0$ are

$$Q_{22} = -g(y_1), Q_{21} = 0, |y_1| \le a; u_2 = 0, Q_{21} = 0, |y_1| > a. \quad (1)$$

Hear y_i is the Cartesian coordinates in the deformed state, Q_{ij} is the component of stress tensor measured per unit area in the deformed state, u_j is the component of the displacement vector.
Introducing the complex variables

$$z_j = y_1 + \mu_j y_2. \quad (2)$$

we may express the stress and displacement components in the terms of analytic functions $\Phi_j(z_j)$ in the following form (the case of non-equal roots $\mu_1 \ne \mu_2$ is considered hear for example, in the terminology of (Guz' A.N. (1983))

$$\begin{aligned}
Q_{22} &= 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)];\\
Q_{21} &= -2\,\mathrm{Re}[\gamma_{21}^{(1)}\mu_1\Phi_1'(z_1) + \gamma_{21}^{(2)}\mu_2\Phi_2'(z_2)];\\
Q_{12} &= -2\,\mathrm{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)];\\
Q_{11} &= 2\,\mathrm{Re}[\gamma_{11}^{(1)}\mu_1^2\Phi_1'(z_1) + \gamma_{11}^{(2)}\mu_2^2\Phi_2'(z_2)];\\
u_k &= 2\,\mathrm{Re}[\gamma_k^{(1)}\Phi_1(z_1) + \gamma_k^{(2)}\Phi_2(z_2)]; k = 1, 2.
\end{aligned} \quad (3)$$

The roots $\mu_k, k = 1,2,$ and the values of coefficients $\gamma_{ij}^{(k)}, \gamma_i^{(k)}, i, j, k = 1,2;$ are determined by the constitutive equations of material (for example, for highly elastic materials ones are determined in accordance with the form of the elastic potential function).

Then, by introducing new analytical function $Z(z_j)$ instead of $\Phi_j(z_j), j = 1,2,$

$$\Phi_1(z_1) = Z_0(z_1); \Phi_2(z_2) = -\frac{\mu_1\gamma_{21}^{(1)}}{\mu_2\gamma_{21}^{(2)}} Z_0(z_2); Z(z_j) = Z_0'(z_j) \quad (4)$$

we eventually obtain the Keldish-Sedov boundary problem for finding in the bottom halfplane the analytical function $Z(z_j)$

$$\begin{aligned}
\mathrm{Re}\,Z(y_1) &= -\frac{g(y_1)}{C^{(1)}}; |y_1| \le a, y_2 = 0;\\
\mathrm{Im}\,Z(y_1) &= 0, |y_1| > a; y_2 = 0.
\end{aligned} \quad (5)$$

In the analogous manner the formulations of the plane problems for isolated lateral shear and longitudinal shear cracks are building (detailed narration see in (Guz' A.N.(1983))).

For near the surface crack as an example we consider the formulation of the axisymmetrical problem *for a halfspace containing a penny-shaped crack under radial shear* (Nazarenko V.M., Bogdanov V.L. and Altenbach H. (2000)).
We will consider a semiinfinite solid containing a penny-shaped crack parallel to the free surface. The crack of the radius a is situated in the upper halfspace $x_3 \geq -h$ in the plane $x_3 = 0$ with the centre on Ox_3-axis (here $x_j, j = 1,2,3$ is the system of Cartesian coordinates referred to the non-deformed state). An initial compression is applied in the Ox_1x_2-plane so that a uniform initial stress and strain state is realized:

$$S_{33}^0 = 0, S_{11}^0 = S_{22}^0 \neq 0,$$
$$u_m^0 = \delta_{jm}(\lambda_j - 1)x_j; \lambda_1 = \lambda_2 \neq \lambda_3, \lambda_j = const.; j,m = 1,2,3. \qquad (6)$$

Here the components of the displacement vector are given by u_j and the components of the symmetric stress tensor in the initial state are S_{ij}^0; u_j^0 are components of the displacement vector corresponding to the initial stress. The superscript '0' denotes the parameters referred to the initial state. The values $\lambda_j, j = 1,2,3,$ denote the extensional (or contractional) ratio along the x_j-axis while δ_{ij} is the Kronecker's symbol.
To describe the actual ('disturbed') state the Kirchhoff nonsymmetric stress tensor t_{ij} (or also in other terms Kirchhoff-Lagrange or 1-st Piola-Kirchhoff stress tensors are in use) is applied here (the components of this stress tensor measured per unit area in the undeformed state).

The boundary conditions on the faces of the crack and on the boundary of the halfspace we may formulate as follows:

$$t_{33} = 0, t_{3r} = -\tau(r) \qquad (x_3 = \pm 0, 0 \leq r < a),$$
$$t_{33} = 0, t_{3r} = 0 \qquad (x_3 = -h, 0 \leq r < \infty), \qquad (7)$$

where r, θ, x_3 are cylindrical coordinates obtained from Cartesian coordinates x_j. In other words, on the crack faces equal and oppositely-directed stresses $\tau(r)$ are applied (antisymmetrically with respect to the plane $x_3 = 0$, radial shear) while the halfspace boundary $x_3 = -h$ is free of stress. It should be noted that the values of stresses $\tau(r)$ are supposed small in comparison with the value of S_{11}^0.
Considering, for example, incompressible solids we can write the linearised equilibrium equation in the displacements u_j in the form

$$\kappa_{im\alpha\beta} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta} + q_{\alpha m} \frac{\partial p}{\partial x_\alpha} = 0,$$
$$q_{m\alpha} \frac{\partial u_\alpha}{\partial x_m} = 0, (i, m, \alpha, \beta = 1,2,3), \qquad (8)$$

while the stress tensor t_{ij} can be obtained from

$$t_{ij} = \kappa_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + q_{ij} p, q_{mk} = \delta_{mk} q_m,$$
$$\kappa_{ij\alpha\beta} = \lambda_j \lambda_\alpha [\delta_{ij}\delta_{\alpha\beta} a_{i\beta} + (1 - \delta_{ij})(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\mu_{ij}] + \delta_{i\beta}\delta_{j\alpha} S_{\beta\beta}^0. \qquad (9)$$

The values $\kappa_{ij\alpha\beta}, \mu_{ij}, a_{ij}$ and q_m depend on material properties (these also result in the linearised constitutive law, that is the relation between Kirchhoff stress tensor t_{ij} and Green strain tensor ε_{ij}, see, for example, in (Guz' A. N., Dyshel M. Sh. and Nazarenko V.M. (1992)).

Following (Guz' A. N. (1999)), by introducing potential harmonic functions

$$\phi_i(r, z_i), z_i = (n_i^0)^{-1/2} x_3, i = 1,2, \tag{10}$$

for *different root* $n_1^0 \neq n_2^0$
or potential harmonic functions

$$\phi(r, z_1), F(r, z_1), z_1 = (n_1^0)^{-1/2}, \tag{11}$$

for *equal roots* $n_1^0 = n_2^0$,
we can obtain the general solutions of Equations (8) in the form
for different roots

$$\begin{aligned}
u_r &= \frac{\partial}{\partial r}(\phi_1 + \phi_2),\\
u_3 &= m_1^0 (n_1^0)^{-1/2} \frac{\partial \phi_1}{\partial z_1} + m_2^0 (n_2^0)^{-1/2} \frac{\partial \phi_2}{\partial z_2},\\
t_{33} &= C_{44}^0 \left(d_1^0 l_1^0 \frac{\partial^2 \phi_1}{\partial z_1^2} + d_2^0 l_2^0 \frac{\partial^2 \phi_2}{\partial z_2^2}\right),\\
t_{3r} &= C_{44}^0 \frac{\partial}{\partial r}\left[(n_1^0)^{-1/2} d_1^0 \frac{\partial \phi_1}{\partial z_1} + (n_2^0)^{-1/2} d_2^0 \frac{\partial \phi_2}{\partial z_2}\right];
\end{aligned} \tag{12}$$

for equal roots

$$\begin{aligned}
u_r &= -\frac{\partial \phi}{\partial r} - z_1 \frac{\partial F}{\partial r},\\
u_3 &= (n_1^0)^{-1/2} (m_1^0 - m_2^0 - 1) F - m_1^0 \Phi - m_1^0 z_1 \frac{\partial F}{\partial z_1},\\
t_{33} &= C_{44}^0 \left[(d_1^0 l_1^0 - d_2^0 l_2^0) \frac{\partial F}{\partial z_1} - d_1^0 l_1^0 \frac{\partial \Phi}{\partial z_1} - d_1^0 l_1^0 z_1 \frac{\partial^2 F}{\partial z_1^2}\right],\\
t_{3r} &= C_{44}^0 \frac{\partial}{\partial r}\left[(d_1^0 - d_2^0) F - d_1^0 \Phi - d_1^0 z_1 \frac{\partial F}{\partial z_1}\right].
\end{aligned} \tag{13}$$

Here

$$\Phi \equiv \frac{\partial \phi}{\partial z_1}, \tag{14}$$

and the values $C_{44}^0, m_i^0, l_i^0, n_i^0$ and $d_i^0 (i = 1,2)$ depend on the initial stresses as far as on the material behaviour model (see details, for example, in (Guz' A. N., Dyshel M. Sh. and Nazarenko V.M. (1992)).
So, using (12) or (13) we can re-formulate boundary problem (7) in the terms of potential harmonic functions (10) or (11) for different or equal roots case correspondently. Such re-formulation then allows to use the integral transformations apparatus (Fourier type for plane and Fourier-Henkel type for space problems).

4. CRITERIA, SOME RESULTS AND CONCLUSION

Griffith-Irwin type fracture criteria

Similar to the classical case when there are no initial stresses acting along the crack (Griffith A.A. (1920)) we also can write for the case under consideration the basic energy conservation equation for crack advance in the analogous form

$$\delta U_0 + \delta A^e_{\delta\Sigma} = 0, \tag{15}$$

where δU_0 - the internal energy which is determined by the surface energy, $\delta A^e_{\delta\Sigma}$ - the energy flux into the crack tip due to the decrease of the strain energy coursed by the crack tip advancing by some value of δl ($\delta\Sigma$ is the increment of the crack surface area; for example, for the crack located in y_1Oy_3-plane:

$$|y_1| \le a, y_2 = \pm 0, -\infty < y_3 < +\infty, \tag{16}$$

measured per unit length along the Oy_3 axis value $\delta\Sigma$ is $\delta\Sigma = 2\delta l$).

Following Irwin **(**Irwin G.R. (1958)) the energy flux $\delta A^e_{\delta\Sigma}$ may be defined through the corresponding components of the stress tensor and displacement vector near the crack tip on the crack prolongation. For example, for the crack case (16) the energy flux is (Guz' A.N. (1992), also see notation of section 3)

$$\delta A^e_{\delta\Sigma} = -\int_0^{\delta l} (Q_{22}u_2 + Q_{21}u_1 + Q_{23}u_3)dx., \tag{17}$$

Similar to the classical case we can determine the stress intensity factors as coefficients with singularities in the corresponding stress components near the tips of the crack, for example for the crack case (16)

$$\begin{aligned} K_I &= \lim_{r\to a}[2\pi(r-a)]^{1/2}Q_{22}(r,0); \\ K_{II} &= \lim_{r\to a}[2\pi(r-a)]^{1/2}Q_{21}(r,0); , \\ K_{III} &= \lim_{r\to a}[2\pi(r-a)]^{1/2}Q_{23}(r,0). \end{aligned} \tag{18}$$

Here r ($r > a, (r-a) << 1$) is the distance from the crack tip in the Oy_1-axis direction.

Finally using formulae type (17), (18) and relation between stress tensor and displacement vector we can obtain the criteria (15) in the next common form as Griffith-Irwin type fracture criteria for material with initial stresses acting along cracks faces

$$C_1^0 K_I^2 + C_2^0 K_{II}^2 + C_3^0 K_{III}^2 = \Gamma^0, \tag{19}$$

where C_1^0, C_2^0, C_3^0 are the coefficients depending on material properties, crack type (form and location) and the values of initial stresses; Γ^0 is material constant defining by material surface energy γ and generally speaking also depending on initial stress values. One or two from three stress intensity factors K_I, K_{II}, K_{III} may be equal to zero. Note that in lack of initial stresses the criteria (19) transfer in the classical fracture criteria of Griffith-Irwin type (i.e., $S^0_{ij} = 0$ and stress tensor Q_{ij} or stress tensor t_{ij} is treated as Cauchy's stress tensor σ_{ij} in the classical theory of elasticity).

Then if crack form and crack location are determined the main problem is the finding of the intensity stress factors K_I, K_{II}, K_{III} for the acting initial stresses S^0_{ij} and verification of the fracture criterion (19).

Some results

We guess it is not worth here to particularize rather complicated mathematical methods of solving the formulated boundary problems (type of (1) or (7)) and the mathematical apparatus used (the very details are given in, for example, Guz' A.N. (1980), Guz' A. N. (1983), Guz' A.N. (1992), Guz' A. N., Dyshel M. Sh. And Nazarenko V.M. (1992), Babich V.M., Guz' A.N. and Nazarenko V.M. (1991), Guz' A.N, Nazarenko V.M. and Nikonov V.A. (1991), Nazarenko V.M., Bogdanov V.L. and Altenbach H. (2000)). It is worth to say only that the mathematical manipulations result in: *for isolated* crack – the Keldish-Sedov problem for the analytical functions in the plane case and the mixed problem for harmonic potential functions in the space case, *for near the surface crack* – the system of Fredholm integral equations.

Hereafter we will enounce essential results with focus on *space axisymmetric problems* and *near the surface crack*. As for the plane problems for isolated crack only the basic conclusions will be written out below (see details in Guz' A.N. (1992)).

Internal penny-shaped cracks.

Isolated normal-rapture crack.

The axisymmetric boundary problem for the crack of radius a located in the $y_3 = 0$-plane as

$$\{0 \le r < a; 0 \le \theta < 2\pi; y_3 = \pm 0\} \tag{20}$$

in r, θ, y_3 (or $z_j \equiv n_j^{-1/2} y_{3,} j = 1,2$) cylindrical coordinates obtained from Cartesian ones $y_1, y_2, y_3,$ is formulated for the upper halfspace $y_3 \ge 0$ as following:

$$\begin{aligned} &Q_{33} = -\sigma_z(r), Q_{3r} = 0 \qquad (0 \le r < a, y_3 = 0) \\ &u_3 = 0, Q_{3r} = 0 \qquad (a < r < +\infty, y_3 = 0) \end{aligned} \tag{21}$$

(we follow Guz' A. N. (1983) and Guz' A.N. (1992) with its accepted notations; $\sigma_z(r)$ is the normal symmetric with respect to $y_3 = 0$-plane loading on the crack faces).

The components Q_{33} and Q_{3r} of the stress tensor Q are determined near the crack tip in its plane by the formulas

$$Q_{33} = [2\pi(r-a)]^{-1/2} K_I \text{ provided } r > a, y_3 = 0, \text{ and } Q_{3r} = 0 \text{ under } y_3 = 0, \tag{22}$$

and the stress intensity factors are

$$\begin{aligned} &K_I = \lim_{r \to a} [2\pi(r-a)]^{1/2} Q_{33}(r,0) = \frac{2}{\sqrt{\pi a}} \int_0^a \frac{r\sigma_z(r)}{\sqrt{a^2 - r^2}} dr, \\ &K_{II} = \lim_{r \to a} [2\pi(r-a)]^{1/2} Q_{3r}(r,0) = 0, \\ &K_{III} = \lim_{r \to a} [2\pi(r-a)]^{1/2} Q_{3\theta}(r,0) = 0. \end{aligned} \tag{23}$$

It is obvious that it the stress intensity factor K_I in this case is not depending on the initial stresses $S_{11}^0 = S_{22}^0$ while the crack faces displacements near the crack tip u_3 is depending on initial stresses as well as coefficient C_1^0 in (19). Displacement u_3 is given as following for equal and unequal root n_1, n_2 cases:

Equal roots $n_1 = n_2$ case

$$u_3 = 2\sqrt{\frac{a-r}{2\pi}} K_I \frac{1 + 2m_1 - m_2}{C_{44}\sqrt{n_1}(l_1 - l_2)(1+m_1)(1+m_2)}, \quad r < a; \tag{24}$$

Unequal roots $n_1 \neq n_2$ *case*

$$u_3 = 2\sqrt{\frac{a-r}{2\pi}} K_I \frac{m_1 - m_2}{C_{44}(l_1\sqrt{n_1} - l_2\sqrt{n_2})(1+m_1)(1+m_2)}, \quad r < a. \tag{25}$$

In (24), (25) the values $C_{44}, m_i, l_i, i = 1,2,$ depend on the initial stresses as far as depend on the material properties.

Near the free surface normal-rapture crack.

The crack of the radius a is situated in the upper halfspace $x_3 \geq -h$ in the plane $x_3 = 0$ with the centre on Ox_3-axis (here $x_j, j = 1,2,3$ is the system of Cartesian coordinates referred to the non-deformed state). Following Babich V.M., Guz' A.N. and Nazarenko V.M. (1991), the axisymmetric boundary problem under initial stresses (6) may be formulated in a manner:

$$\begin{aligned} &t_{33} = \sigma(r), t_{33} = 0 \qquad (x_3 = \pm 0, 0 \leq r < a); \\ &t_{33} = 0, t_{3r} = 0 \qquad (x_3 = -h, 0 \leq r < \infty) \end{aligned} \tag{26}$$

(see notation of section 3).

The mathematical part of investigation was carried out using Henkel's integral transformations along the radial coordinate r .The problem was reduced to a system of paired integral equations and finally using the method proposed by Uflyand Ya.S. (1977) a system of Fredholm's integral equations of the second kind with additional condition was obtained (utilizing the solution of Schlemilch's integral equation):

(we are terminating here by the Unequal roots $n_1^0 \neq n_2^0$ *case only)*

$$\begin{aligned} &f(\xi) + \frac{k_1}{\pi k}\int_0^1 M_1(\xi,\eta) f(\eta) d\eta - \frac{2k_1}{\pi k}\int_0^1 N_1(\xi,\eta) g(\eta) d\eta = \frac{k_1}{\pi k} q(\xi); \\ &g(\xi) + \frac{k_2}{\pi k}\int_0^1 M_2(\xi,\eta) g(\eta) d\eta - \frac{2k_2}{\pi k}\int_0^1 N_2(\xi,\eta) f(\eta) d\eta + const\frac{k_2}{\pi k} = 0; \\ &\int_0^1 g(\xi) d\xi = 0 \qquad (0 \leq \xi \leq 1, 0 \leq \eta \leq 1); \\ &q(\xi) = 4[p(0) + \xi \int_0^{\pi/2} p'(\xi \sin\zeta) d\zeta]; p(\zeta) \equiv \frac{\sigma(a\zeta)}{C_{44}^0 d_1^0 l_1^0}; \\ &k_1 = l_1^0 (n_1^0)^{-1/2}; k_2 = l_2^0 (n_2^0)^{-1/2}; k = k_1 - k_2. \end{aligned} \tag{27}$$

Equations (27) are given in dimensionless form. Unknown constant $const$ is attached to the additional condition (third equation in (27)).

The kernels $M_i(\xi,\eta), N_i(\xi,\eta), i = 1,2,$ of the integral equations are defining as:

$$M_1(\xi,\eta) = R_1(\eta+\xi) - R_1(1+\xi) + R_1(\eta-\xi) - R_1(1-\xi);$$

$$N_1(\xi,\eta) = S_1(\eta+\xi) + S_1(\eta-\xi); \qquad M_2(\xi,\eta) = S_2(\eta+\xi) + S_2(\eta-\xi);$$

$$N_2(\xi,\eta) = R_2(\eta+\xi) - R_2(1+\xi) + R_2(\eta-\xi) - R_2(1-\xi);$$

$$R_1(\zeta) = 2\{2\frac{k_2}{k} I_0(\beta_1+\beta_2,\zeta) - \frac{1}{2}\frac{(k_1+k_2)}{k}[\frac{k_2}{k_1} I_0(2\beta_2,\zeta) + I_0(2\beta_1,\zeta]\};$$

$$S_1(\zeta) = \frac{(k_1+k_2)}{k}\{I_1(\beta_1+\beta_2,\zeta) - \frac{1}{2}[I_1(2\beta_1,\zeta) + I_1(2\beta_2,\zeta)]\}; \qquad (28)$$

$$S_2(\zeta) = 2\{2\frac{k_1}{k_2} I_0(\beta_1+\beta_2,\zeta) - \frac{1}{2}\frac{(k_1+k_2)}{k}[\frac{k_1}{k_2} I_0(2\beta_2,\zeta) + I_0(2\beta_1,\zeta)]\};$$

$$R_2(\zeta) = \frac{(k_1+k_2)}{k}\{I_{-1}(\beta_1+\beta_2,\zeta) - \frac{1}{2}[I_{-1}(2\beta_1,\zeta) + I_{-1}(2\beta_2,\zeta)]\};$$

$$I_0(\rho,\zeta) = \rho(\zeta^2+\rho^2)^{-1}; \qquad I_{-1}(\rho,\zeta) = -\frac{1}{2\beta}\log(\zeta^2+\rho^2);$$

$$I_1(\rho,\zeta) = \beta(\rho^2-\zeta^2)(\zeta^2+\rho^2)^{-2}; \quad \beta = ha^{-1}; \quad \beta_i = \beta(n_i^0)^{-1/2}, i=1,2.$$

Here β is dimensionless distance from crack plane $x_3 = 0$ to the free boundary $x_3 = -h$.

Firstly it should be noted that in the case of $\beta \to \infty$ we obtain the isolated crack. It may be shown that under $\beta \to \infty$ we have $M_1 \to 0, M_2 \to 0, \frac{1}{\beta} N_1 \to 0, \beta N_2 \to 0$, and then

$$\frac{k}{k_1} f(\xi) = \frac{4}{\pi}[p(o) + \xi \int_0^{\pi/2} p'(\xi \sin\zeta) d\zeta; \quad \lim_{\beta\to 0}[\beta g(\xi)] = const; \qquad (29)$$

$const = 0$ (from additional condition),
and finally

$$f(\xi) = \frac{4k_1}{\pi k}\frac{d}{d\xi}\int_0^{\xi}\frac{\eta p(\eta) d\eta}{\sqrt{\xi^2-\eta^2}} \qquad \text{under } \beta \to \infty \qquad (30)$$

(it was taken into account the equality

$$\frac{d}{d\xi}\int_0^{\xi}\frac{\eta p(\eta) d\eta}{\sqrt{\xi^2-\eta^2}} = p(0) + \xi\int_0^{\xi}\frac{p'(\eta) d\eta}{\sqrt{\xi^2-\eta^2}}). \qquad (31)$$

From (27), (28) we can obtain the stress intensity factors K_I and K_{II} in the form:

$$K_I = \lim_{r\to a}[2\pi(r-a)]^{1/2} t_{33}(r,0) = -C_{44}^0 d_1^0 l_1^0 \frac{k\sqrt{\pi a}}{2k_1}\int_0^1 f(\xi) d\xi \, ,$$

$$K_{II} = \lim_{r\to a}[2\pi(r-a)]^{1/2} t_{3r}(r,0) = -C_{44}^0 (n_1^0)^{-1/2} d_1^0 \frac{k\sqrt{\pi a}}{2k_2}\beta\, g(1). \qquad (32)$$

Let us note that according (32) stress intensity factors K_I and K_{II} both are not equal zero and also depend on initial stresses.
If we will take into account for the case $\beta \to \infty$ (isolated crack) the relations (29), (30), we obtain the result

$$K_I^\infty \equiv \lim_{\beta\to\infty} K_I = -\frac{2}{\sqrt{\pi a}}\int_0^a \frac{t\sigma(t) dt}{\sqrt{a^2-t^2}} \quad , \quad K_{II}^\infty \equiv \lim_{\beta\to\infty} K_{II} = 0, \qquad (33)$$

which coincides with one above mentioned (23) accurate to the notation $\sigma_z(r) = -\sigma(r)$. In particular, for the uniform internal pressure

$$\sigma(r) = -p_0 = const \tag{34}$$

the value K_I^∞ is

$$K_I^\infty = \frac{2p_0\sqrt{a}}{\sqrt{\pi}} \tag{35}$$

Numerical examination of the system of equations (27), (28) was carried out by the Boubnov-Galerkin method with the system of orthonormal on the interval $\xi \in [0,1]$ biased Legendre's polynomials $\sqrt{2k-1}P_{k-1}(2\xi - 1), k = 1,2,3 \dots$. Numerical integration was conducted using Gauss quadrature formulas. The highly elastic material with Treloar potential (Treloar L.R.G. (1955)). In Table 1 are given the values K_I / K_I^∞ for dimensionless distance $\beta = h/a = 1.05; 1.30$ and extensional (contractional) ratio λ_1= 1.30 (extension), 0.99 (initial stresses practically are absent), 0.80 (compression). The initial stress $S_{11}^0 = S_{11}^0(\lambda_1)$ depends on ratio λ_1 (or λ_1 depends on S_{11}^0) through the elastic potential function. The results are given for the case of uniform internal pressure (34).

Table 1

β	$\lambda = 1.30$ (extension)	$\lambda_1 = 0.99$ (initial stresses rather small)	$\lambda_1 = 0.80$ (compression)
1.05	1.2014	1.2067	1.9721
1.30	1.1320	1.1327	1.4304

Data in Table 1 shows that for the subsurface crack stress intensity factors essentially depend on the initial stresses. Extension leads to the decrease of K_I while compression increases the value K_I. For the case $\lambda_1 = 0.99$ the results of Table 1 are much closed (disagreement is less then 1 %) to the known results (see Kassir M.K. and Sih G.C. (1975)) for the classical elastic theory case (there are no initial stresses).

Isolated crack under radial shear.

The axisymmetric boundary problem for the crack (20) of radius a is formulated for the upper halfspace $y_3 \geq 0$ as following (Guz' A.N. (1992)):

$$\begin{aligned} &Q_{33} = 0, Q_{3r} = -\tau_{zr}(r) \qquad (0 \leq r < a, y_3 = 0) \\ &u_r = 0, Q_{3r} = 0 \qquad (a < r < +\infty, y_3 = 0) \end{aligned} \tag{36}$$

We assume that on the crack faces equal and oppositely-directed stresses $-\tau_{zr}(r)$ are applied (antisymmetrically with respect to the plane $y_3 = 0$).

The stress intensity factors defined by (23) in the case under consideration are:
(For the cases both equal $n_1 = n_2$ and unequal $n_1 \neq n_2$ roots)

$$K_I = 0; \quad K_{II} = \frac{2}{\sqrt{\pi a}} \frac{1}{a} \int_0^a \frac{r^2 \tau_{zr}(r) dr}{\sqrt{a^2 - r^2}}. \tag{37}$$

Analogously to the case of isolated normal-rapture crack the stress intensity factor K_{II} in this case is not depending on the initial stresses $S_{11}^0 = S_{22}^0$ while the crack faces displacements near the crack tip u_r is depending on initial stresses as well as coefficient C_2^0 in (19).

Near the free surface crack under radial shear.
Rather extensive formulation of the problem is given as an example in Section 3, see (6)-(14). In the result of somewhat analogous mathematical procedure (see Nazarenko V.M., Bogdanov V.L. and Altenbach H. (2000)) as for normal-rapture subsurface crack the next system of the Fredholm's integral equations in dimensionless form was obtained:
(here we will quote the relationships for the Unequal roots $n_1^0 \neq n_2^0$ *case only)*

$$f(\xi)+\frac{4k_1}{\pi k}\int_0^1 f(\eta)K_{11}(\xi,\eta)d\eta-\frac{4k_1}{\pi k}\int_0^1 g(\eta)K_{12}(\xi,\eta)d\eta=0;$$

$$g(\xi)+\frac{4k_2}{\pi k}\int_0^1 f(\eta)K_{21}(\xi,\eta)d\eta-\frac{4k_2}{\pi k}\int_0^1 g(\eta)K_{22}(\xi,\eta)d\eta=\frac{4k_2}{\pi k}\xi\int_0^{\pi/2} q'(\xi\sin\zeta)d\zeta; \quad (38)$$

$$q(\rho)=\frac{\rho\tau(a\rho)}{C_{44}^0(n_1^0)^{-1/2}d_1^0}.$$

The kernels in (38) take the forms

$$K_{11}(\xi,\eta)=\frac{k}{k}[2I_1(\beta_1+\beta_2,\eta)-\frac{(k_1+k_2)}{2k_2}I_1(2\beta_1,\eta)-\frac{(k_1+k_2)}{2k_1}I_1(2\beta_2,\eta)];$$

$$K_{12}(\xi,\eta)=\frac{(k_1+k_2)}{k}\{[\eta^{-1}I_0(\beta_1+\beta_2,\eta)-I_0(\beta_1+\beta_2,1)]-\frac{1}{2}[\eta^{-1}I_0(2\beta_1,\eta)-I_0(2\beta_1,1)]-$$
$$-\frac{1}{2}[\eta^{-1}I_0(2\beta_2,\eta)-I_0(2\beta_2,1)]\};$$

$$K_{21}(\xi,\eta)=-\frac{(k_1+k_2)}{k}\eta[I_2(\beta_1+\beta_2,\eta)-\frac{1}{2}I_2(2\beta_1,\eta)-\frac{1}{2}I_2(2\beta_2,\eta)]; \quad (39)$$

$$K_{22}(\xi,\eta)=-\frac{k_1}{k}\eta\{2[\eta^{-1}I_1(\beta_1+\beta_2,\eta)-I_1(\beta_1+\beta_2,1)]-\frac{(k_1+k_2)}{2k_1}[\eta^{-1}I_1(2\beta_1,\eta)-$$
$$-I_1(2\beta_1,1)]-\frac{(k_1+k_2)}{2k_2}[\eta^{-1}I_1(2\beta_2,\eta)-I_1(2\beta_2,1)]\};$$

where

$$I_0(\rho,\eta)=\frac{1}{4}\log\frac{\rho^2+(\xi+\eta)^2}{\rho^2+(\xi-\eta)^2};$$

$$I_1(\rho,\eta)=\frac{2\rho\xi\eta}{(\rho^2+\xi^2+\mu^2)^2-4\xi^2\eta^2}; \quad (40)$$

$$I_2(\rho,\eta)=-\frac{1}{\rho}I_1(\rho,\eta)[1-\frac{4\rho^2(\rho^2+\xi^2+\eta^2)}{(\rho^2+\xi^2+\eta^2)^2-4\xi^2\eta^2}];$$

and $k, k_i, \beta, \beta_i, i=1,2,$ are determined according (27), (28).

We note from (38) - (40) that the kernels at $\xi=0$ are $K_{ij}(0,\eta)=0, i, j=1,2.$ Besides, the right part of the second equation in (38) is equal to zero at $\xi=0$ then it follows $f(0)=g(0)=0$.

The stress intensity factors (see (32) notation) here are defined by

$$K_I = -C_{44}^0 d_1^0 l_1^0 \frac{k}{2k_1}\sqrt{\pi a}\, f(1);$$
$$K_{II} = C_{44}^0 (n_1^0)^{-1/2} d_1^0 \frac{k}{2k_2}\sqrt{\pi a}\int_0^1 g(\xi)d\xi \,. \quad (41)$$

From the last we can see that, firstly, the stress intensity factor K_I is not zero, secondly, both of the stress intensity factors K_I and K_{II} are effected by the initial stress (6) $S_{11}^0 = S_{22}^0$ (or extension/contraction ratio $\lambda_1 = \lambda_2$), and thirdly, K_I and K_{II} also depend on the distance h (or the dimensionless distance β) from the crack to the free of stresses boundary.

When β tends to the infinity the case of a crack in an infinite material can be obtained. As for the normal-rapture near the surface crack we will show that the stress intensity factors K_I and K_{II} tend the values K_I^∞ and K_{II}^∞ (see notation (33)), coincide with those (37) obtained for an isolated circular crack.

It follows from (39), (40) that under $\beta \to \infty$ the kernels of the Fredholm integral equations tend to zero:

$$\lim_{\beta\to\infty} K_{ij}(\xi,\eta) = 0, \quad i, j = 1,2 \,. \quad (42)$$

Then after some manipulations we seek

$$f^\infty(\xi) = 0;$$
$$g^\infty(\xi) = \frac{4k_2}{\pi k}\frac{d}{d\xi}\int_0^\xi \frac{\eta q(\eta)d\eta}{\sqrt{\xi^2-\eta^2}} \quad (43)$$

and finally

$$K_I^\infty = 0;$$
$$K_{II}^\infty = 2\sqrt{\frac{a}{\pi}}\int_0^1 \frac{\eta^2\tau(a\eta)d\eta}{\sqrt{1-\eta^2}} = \frac{2}{\sqrt{\pi}a^{3/2}}\int_0^a \frac{t^2\tau(t)dt}{\sqrt{a^2-t^2}} \quad , \quad (44)$$

that really coincide with (37) accurate to the notation $\tau(r) = \tau_{zr}(r)$.

In the numerical analysis the Boubnov-Galerkin method has been used. Gaussian-quadrature formulas were utilized for numerical integration. Below we present numerical results for incompressible elastic solids with the Treloar elastic potential (Treloar L.R.G. (1955)), *the unequal roots case*, and with the Bartenev-Khazanovich elastic potential (Bartenev G.M. and Khazanovich T.N. (1960)), *the equal roots case*. Results are given for the case of uniform loading $\tau(r) = \tau = const$.

Treloar potential
The Treloar elastic potential allows the description of neo-Hookean type solids. The values of the stress intensity factors ratio K_{II}/K_{II}^∞ and K_I/K_I^∞ versus λ_1 and β are given in Tables 2 and 3, respectively.

Table 2

λ_1	$\beta = 0.5$	0.75	1.0	1.25	1.5	2.0	3.0	10.0
0.9000	2.1133	1.1443	1.0517	1.0227	1.0111	1.033	1.0005	1.0000
0.9999	1.1337	1.0613	1.0298	1.0152	1.0081	1.0027	1.0005	1.0000
1.1000	1.0647	1.0377	1.0215	1.0123	1.0071	1.0026	1.0005	1.0000

Table 3

λ_1	$\beta = 0.5$	0.75	1.0	1.25	1.5	2.0	3.0	10.0
0.9000	1.1726	0.1693	0.0723	0.0378	0.0217	0.0085	0.0020	0.0000
0.9999	0.1130	0.0767	0.0458	0.0279	0.0175	0.0075	0.0019	0.0000
1.1000	0.0549	0.0493	0.0361	0.0249	0.0169	0.0060	0.0002	0.0000

Besides, for the values $\beta = 0.25, 0.5, 0.75, 1.0$ and 1.25, the dependencies of the stress intensity factor ratio K_{II} / K_{II}^{∞} and K_I / K_I^{∞} versus initial elongation (or reduction) λ_1 ($\lambda_1 > 1$ for tension; $\lambda_1 < 1$ for compression) are shown in Fig.1 and Fig.2, respectively. The curves have vertical asymptotes corresponding to the values of the critical reduction λ_1 obtained for this potential in the problem of fracture in compression along a circular crack parallel to the free boundary of a semiinfinite body (see Nazarenko V.M. (1985)), when the fracture process is initiated by the local instability mechanism.

Bartenev-Khazanovich potential.
The Bartenev-Khazanovich potential describes some grid polymers behaviour. Variations of the stress intensity factor ratio K_{II} / K_{II}^{∞} and K_I / K_I^{∞} with the initial elongation (or reduction) λ_1 for this potential are shown in Figs. 3 and 4, respectively, for the values $\beta = 0.25, 0.5, 0.75, 1.0$ and 1.25. We can see that the values of K_{II} / K_{II}^{∞} and K_I / K_I^{∞} tend to infinity under λ_1 tends to the values of the critical reduction λ_1 which is determined for this elastic potential in the problem of fracture in compression along a circular crack parallel to the free boundary of a semiinfinite body (see Guz’ A.N. and Nazarenko V.M. (1985)).

Isolated crack under torsion.
The axisymmetric boundary problem for the crack (20) of radius a is formulated for the upper halfspace $y_3 \geq 0$ as following (Guz’ A.N. (1992)):

$$Q_{3\theta} = -\tau_{z\theta}(r), \quad 0 \leq r < a, \quad y_3 = 0; \qquad u_\theta = 0, \quad a < r < +\infty, \quad y_3 = 0; \tag{45}$$

(we assume that on the crack faces equal and oppositely-directed stresses $-\tau_{z\theta}(r)$ are applied antisymmetrically with respect to the plane $y_3 = 0$).

The stress intensity factor K_{III} defined by (23) in this case is:
(For the cases both equal $n_1 = n_2$ and unequal $n_1 \neq n_2$ roots)

$$K_{III} = \frac{2}{\sqrt{\pi a}} \frac{1}{a} \int_0^a \frac{r^2 \tau_{z\theta}(r) dr}{\sqrt{a^2 - r^2}}. \tag{46}$$

The stress intensity factor K_{III} in this case also is not depending on the initial stresses $S_{11}^0 = S_{22}^0$ while the crack faces displacements near the crack tip u_θ is depending on initial stresses as well as coefficient C_3^0 in (19).

Near the surface crack under torsion.

The crack of the radius a is situated in the upper halfspace $x_3 \ge -h$ in the plane $x_3 = 0$ with the centre on Ox_3-axis (here $x_j, j = 1,2,3$ is the system of Cartesian coordinates referred to the non-deformed state). Following Guz' A.N, Nazarenko V.M. and Nikonov V.A. (1991), the axisymmetric boundary problem under initial stresses (6) may be formulated as next:

$$\begin{aligned} &t_{3\theta}(r, z_3) = -\tau(r); \quad 0 \le r < a, \quad z_3 = \pm 0; \\ &t_{3\theta}(r, z_3) = 0; \quad 0 \le r < +\infty, \quad z_3 = -h_3; \\ &z_3 \equiv (n_3^0)^{-1/2} x_3, \quad h_3 \equiv (n_3^0)^{-1/2} h. \end{aligned} \tag{47}$$

Following then the mathematical procedure generally analogous above mentioned one for the normal-rupture near the surface crack case we can obtain the Fredholm integral equation of the second kind like this:

$$\begin{aligned} &\omega(\xi) - \frac{1}{\pi}\int_0^1 M(\xi,\eta)\omega(\eta)d\eta = G(\xi); \quad 0 \le \xi \le 1; \quad 0 \le \eta \le 1; \\ &G(\xi) = \frac{2\xi}{\pi}\int_0^{\pi/2} X(a\xi \sin\rho)\sin^2\rho \, d\rho; \\ &X(r) = -\frac{2\tau(r)}{C_{44}^0 (n_3^0)^{-1/2} d_3^0}; \\ &M(\xi,\eta) = 2\beta_3[\frac{1}{2\xi\eta}\log\frac{(2\beta_3)^2 + (\xi-\eta)^2}{(2\beta_3)^2 + (\xi+\eta)^2} + \frac{1}{(2\beta_3)^2 + (\xi+\eta)^2} + \frac{1}{(2\beta_3)^2 + (\xi-\eta)^2}]; \\ &M(0,\eta) = 0, \quad G(0) = 0, \quad \omega(0) = 0; \quad \beta_3 \equiv (n_3^0)^{-1/2}\beta; \quad \beta = h/a. \end{aligned} \tag{48}$$

The stress intensity factor K_{III} is defining according

$$K_{III} = \lim_{r \to a} [2\pi(r-a)]^{1/2} t_{3\theta}(r,0) \quad , \tag{49}$$

in our case of the near the surface crack under torsion is

$$K_{III} = -C_{44}^0 (n_3^0)^{-1/2} d_3^0 \sqrt{\pi a}\,\frac{\omega(1)}{2}. \tag{50}$$

The stress intensity factor K_{III} is depending on the initial stresses as so as depend on the dimensionless distance β from the crack to the free boundary.

As for as the passing to the limit $\beta \to \infty$ when we can obtain the isolated crack case we have

$$\lim_{\beta \to \infty} M(\xi,\eta) = 0, \quad \omega^\infty(\xi) = G(\xi), \tag{51}$$

and finally

$$K_{III}^\infty = \frac{2\sqrt{a}}{\sqrt{\pi}}\int_0^{\pi/2} \tau(a\sin\rho)\sin^2\rho \, d\rho = \frac{2}{a\sqrt{\pi a}}\int_0^a \frac{r^2\tau(r)dr}{\sqrt{a^2 - r^2}} \tag{52}$$

coincide with stress intensity factor (46) obtained for isolated crack if assume $\tau(r) \equiv \tau_{z\theta}(r)$.

Below we present numerical results for incompressible elastic solids with the Treloar elastic potential (Treloar L.R.G. (1955)), *the unequal roots case,* and with the Bartenev-Khazanovich elastic potential (Bartenev G.M. and Khazanovich T.N. (1960)), *the equal roots case.* Results are given for the case of uniform loading $\tau(r) = \tau = const$.

The numerical investigation of the integral equation (48) was carried out with Boubnov-Galerkin method as well as collocation method. In the collocation method the collocation points coincide with the nodes of Gaussian-quadrature formulae. Both methods gave practically identical results.

For the Treloar potential the value β_3 and the kernel $M(\xi,\eta)$ is not depend on the initial stresses as well as function $G(\xi)$. As a result the stress intensity factor K_{III} for this material is also independent of the initial stresses while the displacement u_θ is depending on the initial stresses. The character of changing of the function $\omega(\xi)$ in the interval [0, 1] is given in Fig. 5 for the dimensionless distance from crack plane to the free surface $\beta = 0.05$ and $\beta = 0.25$. The values of $\omega(\xi)$ are normalized by dimensionless value τ / C_{10}, where C_{10} is material constant. Dependence of the stress intensity factor ratio $K_{III} / K_{III}^{\infty}$ versus $\beta = h/a$ is shown in Fig. 6.

For the Bartenev-Khazanovich potential the stress intensity factor ratio $K_{III} / K_{III}^{\infty}$ versus the initial elongation λ_1 ($\lambda_1 < 1$) or reduction ($\lambda_1 > 1$) is given in Fig. 7 for dimensionless distance $\beta = 0.05$ and $\beta = 0.125$. In the case of this potential the stress intensity factor K_{III} is essentially depending on the initial stresses (initial stress $S_{11}^0 = S_{11}^0(\lambda_1)$ or initial elongation/reduction $\lambda_1 = \lambda_1(S_{11}^0)$ through the elastic potential function).

All above mentioned results for internal penny-shaped cracks were given for the case of the axial symmetry. As touching general *non axis-symmetrical case* the reader has an opportunity to be acquainted with the detailed information concerning *isolated penny-shaped internal and external cracks* in the work of Guz' A.N. (1992). As an example below we will consider one such a case for the normal-rupture external penny-shaped crack.

External penny-shaped crack

In this subsection we will deal with the *isolated cracks only* following Guz' A.N. (1992). We assume the crack of radius a is situated in the region

$$a < r < \infty, \quad 0 \le \theta < 2\pi, \quad y_3 = \pm 0 \tag{53}$$

(see the notation above).

For the *normal-rapture crack* the general non axis-symmetric boundary problem for the upper halfspase $y_3 \ge 0$ is formulated as

$$\begin{aligned} &u_3 = 0, Q_{3r} = 0, Q_{3\theta} = 0 \quad \textit{for} \quad 0 \le r < a, 0 \le \theta < 2\pi, y_3 = 0; \\ &Q_{33} = -\sigma_z(r,\theta), \quad Q_{3r} = 0, \quad Q_{3\theta} = 0 \quad \textit{for} \quad a < r < \infty, 0 \le \theta < 2\pi, y_3 = 0. \end{aligned} \tag{54}$$

Here $\sigma_z(r,\theta)$ is the normal load density which applied symmetric regarding the plane $y_3 = 0$. This density can be expressed in the Fourier series form

$$\sigma_z(r,\theta) = \sum_{n=0}^{\infty} p_n(r) \cos n\theta \ . \tag{55}$$

For the *equal* $n_1 = n_2$ *and unequal* $n_1 \neq n_2$ *roots case* the stress intensity factor K_I is

$$K_I = \lim_{r\to a}[2\pi(r-a)]^{1/2} Q_{33}(r,\theta) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} a^{n-1/2}\cos n\theta\,[\int_a^{\infty}\frac{r^{1-n}p_n(r)}{\sqrt{r^2-a^2}}dr]. \qquad (56)$$

For the axisymmetrical case we may assume in (56) $p_0(r) = \sigma_z(r), \quad p_n(r) = 0, n = 1,2,3$ i , so we may obtain for this case

$$K_I = \frac{2}{\sqrt{\pi a}}\int_a^{\infty}\frac{r\sigma_z(r)dr}{\sqrt{r^2-a^2}} \qquad (57)$$

The axisymmetric boundary problem for the *isolated crack under radial shear* is formulated for the upper halfspace $y_3 \geq 0$ as

$$\begin{aligned} &u_r = 0, \quad Q_{33} = 0 \quad for \quad 0 \leq r < a, \quad y_3 = 0; \\ &Q_{33} = 0, \quad Q_{3r} = -\tau_{zr}(r) \quad for \quad a < r < \infty, \quad y_3 = 0. \end{aligned} \qquad (58)$$

The value of the stress intensity factor K_{II} in this case is

$$K_{II} = \lim_{r\to a}[2\pi(r-a)]^{1/2} Q_{3r}(r) = \frac{2a}{\sqrt{\pi a}}\int_a^{\infty}\frac{\tau_{zr}(r)dr}{\sqrt{r^2-a^2}}. \qquad (59)$$

Further, the axisymmetric boundary problem for the *isolated crack under torsion* is formulated for the upper halfspace $y_3 \geq 0$ as

$$\begin{aligned} &u_\theta = 0 \quad for \quad 0 \leq r < a, \quad y_3 = 0; \\ &Q_{3\theta} = -\tau_{z\theta}(r) \quad for \quad a < r < \infty, \quad y_3 = 0 \end{aligned} \qquad (60)$$

The value of the stress intensity factor K_{III} in this case is

$$K_{III} = \lim_{r\to a}[2\pi(r-a)]^{1/2} Q_{3\theta}(r) = \frac{2a}{\sqrt{\pi a}}\int_a^{\infty}\frac{\tau_{z\theta}(r)dr}{\sqrt{r^2-a^2}}. \qquad (61)$$

Internal isolated elliptical crack

Normal-rapture crack

Following Guz' A.N. (1992) and Guz' A.N. and Kluchnikov Yu.V. (1984) we assume that the elliptical crack is located in the $y_3 = 0$ plane

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} < 1, \quad y_3 = \pm 0, \quad y_1 = a\cos\phi, \quad y_2 = a\sin\phi. \qquad (62)$$

Here ϕ is parameter angle; a and b is the ellipse semi-major axis the ellipse semi-minor axis accordingly. The symmetric form of the elliptical coordinates (ξ,ξ,η) (see Kassir M.K. and Sih G.C. (1975)) will be used so that in the crack plane $y_3 = 0$ the value $\xi = 0$ means the internal points in the ellipse while the value $\eta = 0$ means the external points. The boundary problem for the upper halfspace $y_3 \geq 0$ has the form

$$\begin{aligned} &Q_{33} = p(y_1, y_2), \quad Q_{31} = 0, \quad Q_{32} = 0 \quad for \quad \xi = 0; \\ &u_3 = 0, \quad Q_{31} = 0, \quad Q_{32} = 0 \quad for \quad \eta = 0. \end{aligned} \qquad (63)$$

Normal load $p(y_1, y_2)$ is applied symmetrically regarding the $y_3 = 0$ plane.

For the simplest case of constant load

$$p(y_1, y_2) = p_{00} = const \qquad (64)$$

the stress intensity factor K_I is defining as

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}Q_{33}; \quad K_I(\phi) = \frac{\sqrt{\pi}\, p_{00}}{E(k)}\sqrt{\frac{b}{a}}\Phi^{1/4}, \quad \Phi \equiv a^2\sin^2\phi + b^2\cos^2\phi, \qquad (65)$$

where the value r is the distance from crack contour along the normal direction and $E(k), \quad k=\sqrt{1-b^2/a^2}$ is the elliptical integral of the second kind.

Crack under shear

For the elliptical crack located as (62) under shear the boundary problem for the upper halfspace $y_3 \geq 0$ is

$$\begin{aligned} &Q_{33}=0, \quad Q_{31}=q_1(y_1,y_2), \quad Q_{32}=q_2(y_1,y_2) \quad for \quad \xi=0; \\ &Q_{33}=0, \quad u_1=0, \quad u_2=0 \quad for \quad \eta=0. \end{aligned} \qquad (66)$$

The tangent loads q_1 and q_2 are applied to the crack faces antisymmetrically regarding the $y_3=0$ plane.

In the case of the uniform shear with density q_0 acting at an angle β with an ellipse semi-major axis the stress intensity factors K_{II} and K_{III} are

$$\begin{aligned} &K_{II} = \lim_{r\to 0} Q_{3n}; \quad K_{II}(\phi) = 4\sqrt{\pi}\,\mu_1(ab)^{-3/2}\Phi^{-1/4}(aC\sin\phi bB\cos\phi); \\ &K_{III} = \lim_{r\to 0} Q_{3t}; \quad K_{III}(\phi) = -4\sqrt{\pi}\,\mu_1(1-\nu_1)(ab)^{-3/2}\Phi^{-1/4}(aB\sin\phi - bA\cos\phi). \end{aligned} \qquad (67)$$

Here μ_1 and ν_1 are the values depending on the material properties and the initial stresses; sub indices n and t denote the direction along external normal and tangent to the ellipse; and C, B are calculated as

$$\begin{aligned} &B = \frac{1}{4\mu_1}ab^2k^2q_0\cos\beta[(k^2-\nu_1)E(k)+\nu_1(1-k^2)K(k)]^{-1}; \\ &C = \frac{1}{4\mu_1}ab^2k^2q_0\sin\beta\{[k^2+\nu_1(1-k^2)]E(k)-\nu_1(1-k^2)K(k)\}^{-1}, \end{aligned} \qquad (68)$$

where $k=\sqrt{1-b^2/a^2}$ and $K(k), E(k)$ are the elliptical integrals of the first and second kind respectively.

As the formulae (67), (68) show for the elliptical crack under shear the stress intensity factors K_{II} and K_{III} for the longitudinal and lateral shear are depending on the initial stresses.

Conclusion

The above results as well as some results of Guz' A.N. (1992) not covered in this paper allow to make the following conclusions.

I. *About the 'resonance' type phenomenon at the approach to the critical value of the initial stresses in compression.*

The of the stress intensity factors increase abruptly when the compressive initial stresses (or the initial reduction ratios) tends to the values corresponding to the local instability loss in compression for the body of the same geometry (i.e. for example for a above mentioned near the surface crack it means the local loss of stability of a semiinfinite solid containing a near the surface crack under the acting of the compressive initial stresses, etc.) .

II. *Singularity type in stresses at the crack tip under initial stresses.*
The order of singularity of the stress redistribution near the tip of the crack coincides with analogous result of the classical linear mechanics of brittle fracture for the all cases examined.

III. *Influence of compression (reduction) and extension (elongation) on the stress intensity factors.*
As a rule compression in the initial stresses (like above mentioned $S_{11}^0 < 0$) or reduction in the initial reduction ratio (like $\lambda_1 < 1$) leads to the increase in absolute values of the stress intensity factors. The exception of this rule is the case of combine loading (i.e. for example the crack under coactions of the longitudinal and lateral shear, etc.), when one of the stress intensity factors can increase while the other can decrease at the same time.

IV. *Isolated cracks.*
As a rule for the 'pure' loading (i.e. under acting of the 'pure' normal load for the normal-rapture crack, 'pure' longitudinal shear loading for the crack under longitudinal shear, etc.) the only stress intensity factor not equal zero is not depending on the initial stresses. While the displacements of the crack faces as well as coefficients in the Griffith-Irwin fracture criteria (like $C_i^0, i = 1,2,3$ in (19)) are depending on the initial stresses.

In the case of the combine loading or when the 'pure' loading is impossible due to the crack shape (for example see the elliptic crack under shear) the stress intensity factors are depending on the initial stresses.

V. *Near the surface cracks.*
Similar to the classical case (with absence of initial stress) the presence of a free boundary in a solid with initial stress as a rule leads to the appearance of two nontrivial stress intensity factors even for the 'pure' loading (see above). For example, in the case of normal-rapture penny-shaped crack both stress intensity factors K_I and K_{II} are not equal zero.

The stress intensity factors are depending on the initial stresses (it seems the only exclusion gives the penny-shaped crack under 'pure' torsion for the Treloar elastic potential due to specific material properties in this case).

When the distance between the crack plane and the free surface tends to infinity the obtained results fully coincides with the results for the isolated crack.

VI. *When the initial stresses are absent.*
When we assume the initial stresses S_{ij}^0 are absent all the above mentioned results are transformed into the classical results for a solid without initial stresses (then we also assume that the stress tensors like above mentioned Q_{ij} or t_{ij} to be the Cauchy's stress tensor σ_{ij} in the classical theory of elasticity).

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